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"Link-homotopie" en topologie de base dimension

Présentée et soutenue par EMMANUEL GRAFF

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‘Link-homotopy’ in low-dimensional topology

Emmanuel Graff

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Introduction

The origins of *knot theory*, which is the main topic of this thesis, are probably to be found in the work of C. F. Gauss 1833 in electrodynamics. There, he defined the number of 'intertwinings' of two trajectories and showed how this number, that we call today the *linking number*, can be computed by a double integral. Several decades later two physicists, W. Thomson (lord Kelvin) and P. G. Tait, set the foundations of knot theory, see [Sil06]. The former proposed a model of matter in which atoms are represented by knot-shaped vortices, the type of the knot determining the atom's physico-chemical properties. To understand matter, it was therefore necessary to classify knots. This initiated the work undertaken by Tait. He provided the first attempt of a classification of knots with less than ten crossings. It was H. Poincaré at the end of the 19th century, who provided, the formal framework for the study of knots, with the development of algebraic topology [Poi95].

Knot theory has the advantage of being inspired by real-life objects. It is the study of knots as they are commonly understood: a piece of string tied in space. The two ends of the string are glued together, so that the resulting knot cannot in general be trivially untied. We then seek to understand the topology of the knot without worrying about its physical characteristics: length, strength, nature of the string, etc. More rigorously, a *knot* is defined as a *smooth embedding* of the circle in the three-dimensional ball. The simplest knot of all, pictured in Figure 1, is just the unknotted circle, which we call the *unknot* or the *trivial knot*. The next simplest knot is called the *trefoil knot*, illustrated in Figure 2.

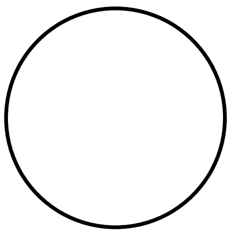


Figure 1: The unknot.

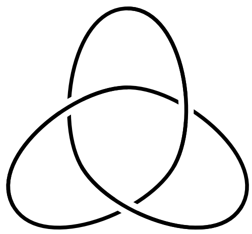


Figure 2: The trefoil knot.

We typically consider and study knots up to several kind of deformations (see below), for which two questions naturally arise. If we give ourselves a knot, can we untie it? If we give ourselves two knots, are they equivalent up to deformation? To answer these questions, we use the notion of knot *invariants*. An invariant is a quantity (number, matrix, polynomial, etc.) associated to each knot, such that if two knots are equivalent, the associated quantities are the same. In practice, it is the contrapositive of this proposition that is used, i.e., if two knots do not have the same invariant, then these knots are not equivalent up to the considered type of deformation.

The first and main type of deformation considered by topologists is the notion of *isotopy*. We say that two knots are isotopic if they are related by an *ambient isotopy* of the three-ball. This transformation corresponds to manipulations that do not involve cutting or passing the string through itself. Figure 3 gives an illustration of an ambient isotopy of the unknot. To date, no isotopy knot

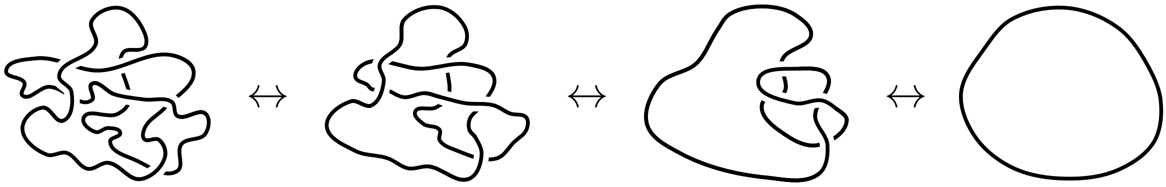


Figure 3: Ambient isotopy untying a tangled unknot.

invariant is really satisfactory. They are generally *incomplete* in the sense that some non-isotopic knots have the same invariant values. This is definitely the case for well-known invariants such as the *crossing number*, the *unknotting number*, the *genus*, the *Jones polynomial* or the *Alexander polynomial*. For some others, whether or not they are complete remains an open question. It is for instance the case for the family of *finite-type* invariants. Finally, the *fundamental group of the knot complement*, when endowed with the *peripheral structure*, forms a complete invariant [Wal68]; but this invariant is difficult to handle, and determining whether two groups are isomorphic is no easy matter either.

Another equivalence relation that later interested knot theorists is *concordance*, initially defined in [FM66]. Two knots are concordant if they co-bound a cylinder smoothly and properly embedded in $B \times [0,1]$, with B the three-dimensional ball, each knot lying respectively in $B \times \{0\}$ and $B \times \{1\}$. Isotopy implies concordance, it is therefore a more permissive notion, and a priori simpler to study. To illustrate our point, let us present the *connected sum* operation on knots (more precisely, we consider here *oriented* knots). Given two knots, we define their connected sum by removing a small arc from each knot and then connecting the four endpoints two by two as in Figure 4. We stress that the

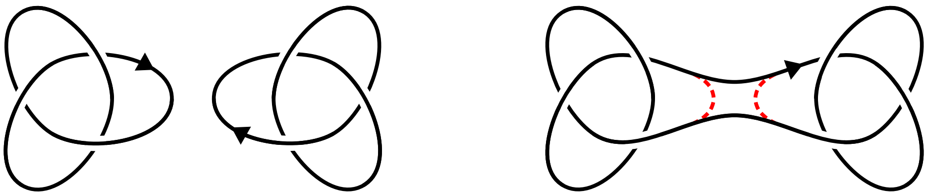


Figure 4: The connected sum of two trefoil knots.

connected sum endows the set of knots up to concordance with an abelian group structure, whereas up to isotopy we only obtain an abelian monoid. However, concordance is still very hard to study, and is still poorly understood. In fact, we do not even know how to determine whether a knot is trivial up to concordance, a property that qualifies it as *slice* knot. This question is the subject of R. H. Fox's famous ribbon/slice conjecture [Fox62]. This conjecture arose from the observation that any *ribbon knot*, i.e., a knot bounding an immersed disk that admits only ribbon singularities, is always concordant to the trivial knot.

Finally, let us consider *link-homotopy*, another type of deformation central to our study. It is a more permissive equivalence relation than the previous two, in the sense that concordance (and therefore isotopy) implies link-homotopy. Link-homotopy was first studied in 1954 by J. W. Milnor in [Mil54]. It is an equivalence relation on *links* (embedding of several circles, called *components*), that allows continuous deformations during which two distinct components remain disjoint at all times, but each component may self-intersect. We give in Figure 5 an example of a link-homotopy; the first deformation in the figure is a *self-crossing change*, a local move that generates link-homotopy. Any

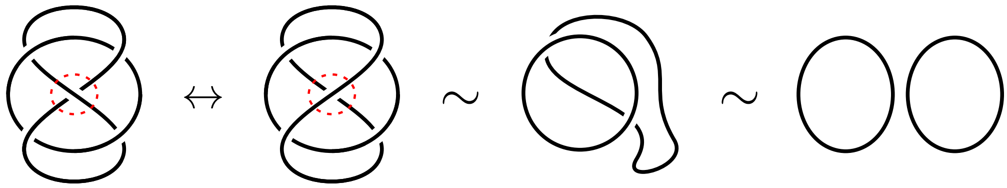


Figure 5: The Whitehead link is trivial up to link-homotopy.

knot is link-homotopic to the trivial one, but for links with more than one component this equivalence relation turns out to be quite rich and intricate. Since J. W. Milnor's seminal work, link-homotopy has been the subject of numerous works in knot theory see e.g., [Gol74, Lev88, Orr89, HL90], but also more generally in the study of co-dimension 2 embeddings (and in particular *knotted surfaces* in dimension 4) [MR85, BT99, AMW17] and *link-maps* (self-immersed spheres) [FR86, Kir88, Kos90, ST19]. In this manuscript, we are interested in the study of link-homotopy for various objects of low-dimensional topology: *braids* and *links* in the classical and *welded* cases. We will also investigate the notion of link-homotopy for *homology cobordisms.*

The following paragraphs provide an overview of our work and its historical context. The content of the thesis will be briefly outlined, along with the main results. Then, at the end of this introduction, the precise structure of the various chapters will be presented.

Braids are ubiquitous objects that can be considered and defined from several points of view. We recall here their geometrical definition due to E. Artin in [Art25]. Let us take a 2-dimensional disk D and let us also take n aligned points $p_1, \dots, p_n$ in the interior of D. An *n-strand* braid $\beta = (\beta_1, \dots, \beta_n)$ is a smooth and proper embedding:

$$(\beta_1, \dots, \beta_n) : \bigsqcup_n [0,1] \to D \times [0,1]$$

satisfying two conditions. Firstly, there exists an n-permutation π, such that for any integer i, the endpoints satisfy $\beta_i(0) = (p_i,0)$ and $\beta_i(1) = (p_{\pi(i)},1)$. Secondly, for any $t \in [0,1]$, the slice $D \times \{t\}$ intersects β in exactly n points, see Figure 6. A braid is said to be *pure* if its associated permutation π is the identity.

E. Artin's work focused mainly on braids up to isotopy (note that, in the context of braids, the ambient isotopies are required to fix the boundary). In [Art47], he describes precisely the *braid group*. This is the group obtained by endowing the set of braids up to isotopy with the braid *composition*, an operation illustrated in Figure 7 which consists in stacking braids on top of each other. In addition, he shows that the braid group acts faithfully on the fundamental group of the punctured disk $D \backslash \{p_1, \cdots, p_n\}$. From this action stems a representation, known as the *Artin representation.* This

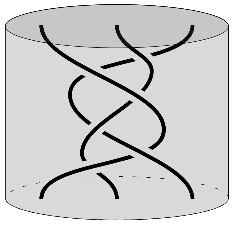


Figure 6: Example of a 3-strand pure braid.

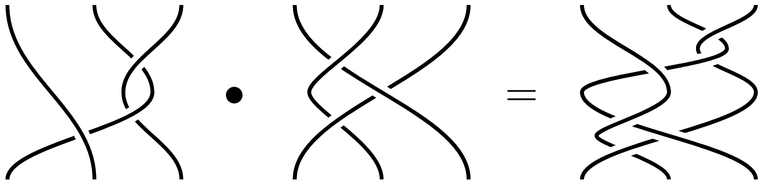


Figure 7: Composition of two braids.

representation has since been declined in various settings and is still being studied today. Finally, E. Artin in [Art47] was the first author to mention the notion of link-homotopy in the context of braids. He raises the question of whether the notions of isotopy and link-homotopy of braids are different.

In [Gol74] D. L. Goldsmith answers the question, giving an example of a non-trivial braid up to isotopy that is trivial up to link-homotopy, see Figure 8. She also gives a presentation of the *homotopy braid group*, i.e., the group of braids up to link-homotopy with braid composition, which appears as a quotient of the classical braid group.

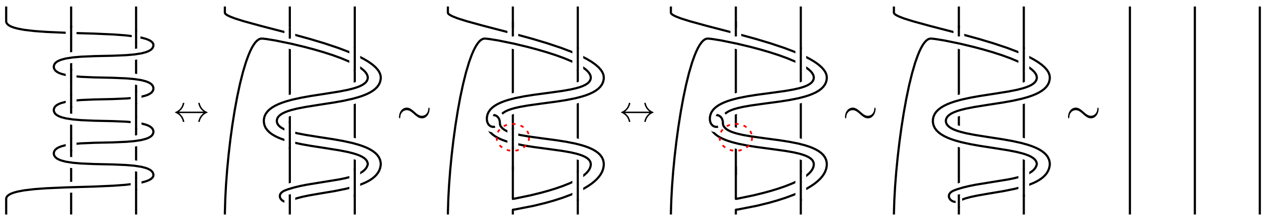


Figure 8: D. L. Goldsmith's example of a braid that is trivial up to link-homotopy, but non-trivial up to isotopy.

Motivated by the 'torsion problem' (see below), S. P. Humphries further pursued the study of braids up to link-homotopy. He defined in [Hum01] a linear representation of the homotopy braid group. However, this representation is not faithful. In contrast, we obtain the following.

Theorem A. *There exists a faithful linear representation of the homotopy braid group for any number of strands.*

We give the detailed definition in Section 2.3; roughly speaking, this representation can be thought of as the 'linearization' of the Artin representation.

Let us state now the *torsion problem*.

The torsion problem. *Is there torsion in the homotopy braid group?*

This problem was first investigated by S. P. Humphries, who showed in [Hum01] that for less than 6 strands, the homotopy braid group is torsion-free. The torsion problem also appears in [BVW22], where the authors mention the more general question of V. Lin, formulated in [Lin96] and taken up in the Kourovka notebook [MK14]: 'Is there a non-trivial epimorphism of the braid group onto a non-abelian group without torsion?'. P. Linnell and T. Schick in [LT07] provide a complete solution by showing that the braid group is *residually torsion-free nilpotent-by-finite*, hence in particular has plenty of non-trivial torsion-free quotients. However, they only give an existence proof, and explicit examples are not known for more than 6 strands. Our second main result on the homotopy braid group solves the torsion problem:

Theorem B. *The homotopy braid group is torsion-free for any number of strands.*

We tackle this problem in two stages. We first prove a *weak* version, by showing that the homotopy braid group is torsion-free for 10 strands or less, using purely classical techniques of braid theory. We then extend this result, proving the statement for any numbers of strands, by using the broader context of *welded* braids (see below). Interestingly, both proofs are based on similar techniques. However, if we restrict ourselves to the case of classical braids, we obtain only a partial result (namely the above weak version). Hence, Theorem B can be seen as one of the few known topological application of the welded (and virtual) knot theory; see for instance [GPV00, ABMW17a, AM19, MY22].

Furthermore, as a corollary of Theorem B, we obtain that the braid group is torsion-free for any number of strands (Corollary 4.3.10). This is a well-known fact due to Fadell and Neuwirth in [FN62, Theorem 8]. Another classical proof of this result is based on a stronger property, shown by P. Dehornoy in [Deh94], which states that braid groups are left-orderable. The property of left-orderability for the homotopy braid group is not known to this day and constitutes an interesting open question, as discussed in Remark 4.3.11.

Finally, the pure homotopy braid group has been studied by N. Habegger and X.-S. Lin in [HL90] as an intermediate object for the classification of links up to link-homotopy. They use the notion of *reduced free group*, which is the quotient of the free group in which each generator commutes with any of its conjugates, a notion due to Milnor [Mil54].

We next address the problem initially posed by J. W. Milnor in [Mil54], of classifying links in the 3-sphere up to link-homotopy. J. W. Milnor himself answered the question for the 2 and 3-component cases. Furthermore, N. Habegger and X.-S. Lin [HL90] proposed a complete classification, using a subtle algebraic equivalence relation on pure braids, where two equivalent braids correspond to link-homotopic links. This classification result remains however somewhat non-effective, owing to this intricate equivalence relation involved. A more direct algebraic approach had been proposed by J. Levine [Lev88] just before the work of N. Habegger and X.-S. Lin in the 4-component case. Our main result concerning links is a new geometric proof of J. Levine's classification of 4-component links up to link-homotopy. Concretely, this accounts to make completely explicit, in a geometric way, the algebraic ingredients used in Habegger–Lin's work, thus providing an effective classification result. This also provides a geometric interpretation of Levine's work. The result can be roughly formulated as follows, see Theorem 3.2.3 for a precise statement.

Theorem C. *There is a complete classification of links up to link-homotopy for less than 4 components, by computable numerical invariant.*

Our approach seems to apply, at least in principle, to links with a higher number of components; we illustrate this with the case of *algebraically-split* 5-component links (that is, 5-component links with vanishing linking numbers). As a matter of fact, the general 5-component case has since been treated independently using our approach by Y. Kotorii and A. Mizusawa in [KM22]. The central tool for our geometric proof is the theory of *claspers*.

The notion of claspers was developed by K. Habiro in [Hab00b], and independently by M. Goussarov in [Gou99, Gou01] in the context of three-manifolds. These are thickened graphs in three–manifolds with some additional structure, on which surgery operations can be performed. They can be use effectively to study knotted objects and their invariants; see for example [Hab00b, Yas09, MY12]. In [Hab00b], K. Habiro describes the *clasper calculus* up to isotopy, which is a set of geometric operations on claspers that yield isotopic surgery results. In particular, he showed the close relationship between claspers and the theory of finite type invariants (also known as Vassiliev invariant). It is well known to experts how clasper calculus can be refined for the study of knotted objects up to link-homotopy (see for example [FY09, Yas09]). This *homotopy clasper calculus*, which we review in Section 1.1.2, is a central tool in our work on both links and braids.

Other important objects of this thesis are *welded braids*. Roughly speaking, welded braids are generalized braid diagrams, where *virtual crossing* are allowed in addition to the classical ones, regarded up to certain local deformations generalizing the usual Reidemeister moves. An example of a welded braid is given in Figure 9, where virtual crossings are represented by transverse double points. As

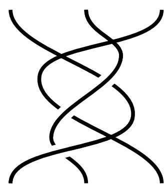


Figure 9: Example of a 3-strand pure welded braid.

with classical braids, welded braids can be endowed with a group structure, resulting in the *welded braid group*, which was first introduced by R. Fenn, R. Rim'anyi, and C. Rourke, in [FRR97]. This group turns out to have several equivalent definitions, of rather different natures, and appears in various contexts under different names. A. G. Savushkina defines it in terms of automorphism of the free group in [Sav96]; the *pure welded braid group* appears as the so-called McCool group in this setting [McC86]. Other authors define it in terms of motion group of circles: J. C. Baez, D. K. Wise, and A. S. Crans [BWC07] call it *loop braid group*, while in [BH13], T. E. Brendle and A. Hatcher call it the *group of untwisted rings*. Finally, welded braids can also be seen as certain cylinders properly embedded in the four-dimensional ball, see for instance [ABMW17a]. We will not discuss here these different points of view, but we refer the reader to C. Damiani's survey [Dam17] for more details. We shall rather focus on the notion of link-homotopy for virtual and welded objects in this context.

H. A. Dye and L. H. Kauffman in [DK10] gave a first definition of link-homotopy for virtual objects in terms of *self-crossing change*, which proves somewhat unsatisfactory (for example, virtual knots are not always trivial up to self-crossing change). Subsequently, B. Audoux, P. Bellingeri, J.-B. Meilhan and E. Wagner in [ABMW17a] and [ABMW17b] defined what appears to be the correct notion of link-homotopy in the welded context in terms of *self-virtualization*. The authors, in light of N. Habegger and X.-S. Lin [HL90], give a correspondence between pure welded braids up to link-homotopy and conjugating automorphisms of the reduced free group. These results have been then extended by

J. Darné in [Dar23], who gave a presentation of the pure welded braid group up to link-homotopy. Based on this presentation and using the technology of *arrow calculus*, we obtain new presentations of welded braid groups up to link-homotopy. Here, the notion of arrow calculus, developed by J.-B. Meilhan and A. Yasuhara in [MY19], is the analogue of claspers calculus in the welded framework. We also use it to extend our linear representation of Theorem A to the *homotopy welded braid group*. Finally, we return to the torsion problem from the welded point of view. We show that the homotopy braid group is torsion-free for any number of strands (Theorem B), thus giving an explicit solution to V. Lin's question.

The last objects discussed in this manuscript are *homology cobordisms*. These are 3-dimensional manifolds that co-bound a surface and induce isomorphisms at the homology level. In the early 2000s, M. Goussarov in [Gou99, Gou01] and K. Habiro in [Hab00b] defined these objects independently, along with the associated clasper calculus as an important class of objects in the theory of finite type invariant of 3-manifolds. Subsequently, N. Habegger, J. Levine and S. Garoufalidis in [Hab00a, Lev01, GL05] studied homology cobordisms as an enlargement of the *mapping class group*. We refer the reader to survey [HM12] for a precise description of these works.

The question of link-homotopy in this context is motivated by the so-called 'Milnor-Johnson correspondence' which draws a strong analogy between braids, *string-links*, concordance, *Milnor invariants* on one hand, and *mapping class groups*, homology cobordisms, *homology cobordism classes*, and *Johnson homomorphisms* on the other hand. We first observe that the natural algebraic approaches to this question do not yield a satisfactory theory. Thus leads us to consider a *graph-claspers*-based definition instead. We explain, based on several counterexamples, how we are naturally led to a definition which, although seemingly rigid, appears to be a promising candidate for a theory of link-homotopy for homology cylinders.

This thesis consists of 5 chapters. Let us outline a bit more precisely the content of each.

Chapter 1 contains the topological and algebraic prerequisites that we will be using throughout the thesis. In Section 1.1, we review the homotopy clasper calculus: after briefly recalling from [Hab00b] K. Habiro's clasper theory, we recall how a fundamental lemma from [FY09], combined with K. Habiro's work, produces a set of geometric operations on claspers having link-homotopic surgery results. In Section 1.2 we introduce the *reduced quotient* of a group and study mainly that of the free group. We prove, in Theorem 1.2.10, the existence and the unicity of a *normal form* for any element of the reduced free group as a product of well-chosen *commutators*.

Chapter 2 is dedicated to the study of braids up to link-homotopy. We start by reinterpreting braids in terms of claspers. In Section 2.1 we define *comb-claspers*, a family of claspers corresponding to braid commutators. They are next used to define a *normal form* on braids up to link-homotopy, thus allowing us to rewrite any braid as an ordered product of comb-claspers. In Section 2.2, we give presentations of homotopy braid groups (Theorem 2.2.1 and Corollary 2.2.6), using the work of [Gol74] and [MK99] as well as the technology of claspers. In Section 2.3, we define and study the representation of the homotopy braid group of Theorem A. We give its explicit computation in Theorem 2.3.5 (see also Example 2.3.7 for the 3-strand case) and show its injectivity in Theorem 2.3.11. Moreover, from the injectivity of the representation follows the uniqueness of the normal form and thus the definition of the *clasp-numbers*, a collection of braid invariants up to link-homotopy. In Section 2.4, we address the torsion problem in the homotopy braid group. Thanks to clasper calculus and a refinement, up to conjugation, of the normal form, we exhibit a potential torsion candidate. We then show that its clasp-numbers must verify a certain equality for it to be a torsion element (see Lemma 2.4.15). Then, in Theorem 2.4.19, we test the equality with the previously defined representation, showing the absence of torsion for 10 strands or less. The proof is based on a computer program (available on [Gra22]), so we can improve the result by optimizing the program or using greater computing power; but this method will always yield a partial result. However, as mentioned above, the ideas in this section combined with welded tools provide a complete answer to the torsion problem.

Chapter 3 focuses on the study of links up to link-homotopy. The method used is based on the precise description of some operations, which generate the algebraic equivalence relation mentioned above in the classification result of N. Habegger and X.-S. Lin [HL90]; we provide them with a topological description in terms of claspers. This new point of view allows us, for a small number of components, to describe when two braids in normal form have link-homotopic closures. We translate in terms of *clasp-number variations* the action of those operations on the normal form. In this way, we recover the classification results of J. W. Milnor [Mil54] and J. Levine [Lev88] for 4 or less components (Theorem C). Moreover, we also classify 5-component algebraically-split links up to link-homotopy (Theorem 3.2.6).

Chapter 4 deals with the study of welded objects. General definitions are first given in Section 4.1, including a review of the arrow calculus developed in [MY19]. Then, in Section 4.2, building on the work of Chapter 2, we show analogous results in the welded context. We give in Theorem 4.2.15 and Corollary 4.2.16 presentations of homotopy welded braid groups, using the work of [Dar23] and [Dam17] as well as arrow calculus. We also show that the linear representation of Theorem A extends to the group of homotopy welded braids. We give its explicit computation in Theorem 4.2.28 and show its injectivity in Theorem 4.2.34. Finally, Section 4.3 returns to the torsion problem. We recast the techniques of Section 2.4 in the larger welded setting using arrow calculus. This allows us to show in Lemma 4.3.5 that the torsion problem is equivalent to whether a given welded braid is conjugate to a classical braid up to link-homotopy. However, using algebraic techniques, we show that such conjugate do not exist, which implies the absence of torsion in the homotopy braid group for any number of strands, as stated in Theorem B.

The final exploratory chapter 5 deals with the study of homology cobordisms over a once-bordered surface Σ. We aim to reinterpret the notion of link-homotopy for these objects. Our initial approach, in Section 5.2, is algebraic in nature and aims to define an action of homology cobordisms on an appropriate 'reduced' quotient of the fundamental group of Σ. However, this action cannot be defined using Milnor's notion of reduced quotient (Counter-examples 5.2.1 and 5.2.2). In Section 5.2.2, we attempt to restrict the action to a larger quotient, namely the *fully reduced quotient*, but this quotient turns out to be too coarse, as illustrated by Theorem 5.2.8. Next, we explore a new approach to defining link-homotopy in terms of graph-claspers in Section 5.3. We explain how this boils down to defining a notion of repetition on leaves (analogous to Lemma 1.1.10). In Section 5.3.2.1, an initial naive definition of repetition is proposed, but it proves unsatisfactory, as illustrated by Example 5.3.11. Finally, in Section 5.3.2.2, a less intuitive definition is suggested. Although we do not delve deeper into the study of this notion within this manuscript, we consider it as a potential avenue for future research.

It should be noted that the results of the first three chapters are essentially contained in the publication [Gra23]. These three chapters however contain more material than [Gra23], including in particular our first (partial) solution to the torsion problem.

Chapter 1

Requirements

In this chapter, we give the basic topological and algebraic tools that will be used throughout the document. In Section 1.1, we define *tangles*. They encompass the objects that we will study in the following sections: braids, string-links, knots and links. We also define *claspers*, powerful topological tools which are particularly well-suited for the study of link-homotopy. Then, in Section 1.2, we turn our attention to the *reduced quotient* of a group. More specifically, we study the *reduced free group*, for which we propose a *normal form* as a product of well-chosen *commutators*.

1.1 Tangles and claspers

Clasper calculus has been developed by K. Habiro in [Hab00b] in the context of *tangles* up to *isotopy* (Definition 1.1.1). Claspers turn out to be in fact a powerful tool to deal with *link-homotopy* (Definition 1.1.2). In Section 1.1.1 we define the main objects and their associated vocabulary. Then we describe in Section 1.1.2 how to handle claspers up to link-homotopy.

1.1.1 General definitions

For simplicity, we decide to define and study *tangles* in the 3-dimensional ball. However, the results presented in this section are naturally adaptable to the study of tangles in any 3-dimensional manifold.

Definition 1.1.1. *An n-*component **tangle** *is a smooth embedding of an n-component, ordered, and oriented* 1*-manifold (a disjoint union of circles and intervals) in the* 3*-dimensional ball. We also required the embedding to be* proper*, which mean that the boundary of the* 1*-manifold must be sent to the boundary of the* 3*-ball. We often identify the tangle with its oriented image (the orientation is induced by the embedding). Each of the embedded component is called a* component *of the tangle.*

Two tangles are **isotopic** *if they are related by an ambient isotopy of the ball, fixing its boundary.*

Definition 1.1.2. *Two tangles are* **link-homotopic** *if there is a homotopy between them fixing the boundary, and such that distinct components remain disjoint during the deformation.*

Remark 1.1.3. *Tangles are faithfully represented by a generic planar projection; generically, the intersection points will not be more than double. By specifying at each crossing which strand passes over the other, and specifying the orientation of the components, we get a* tangle diagram*.*

Theorem 1.1.4. *[AB26, Rei26] Two tangles are isotopic, if and only if, their diagrams are related by a sequence of* Reidemeister moves *(see Figure 1.1) and planar isotopies.*

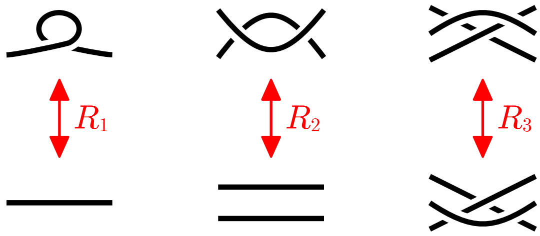


Figure 1.1: The Reidemeister moves.

In the following theorem, we recall an alternative characterization from [Mil54] of the link-homotopy in terms of diagrams.

Theorem 1.1.5. *Two tangles are link-homotopic, if and only if, their diagrams are related by a sequence of Reidemeister moves (see Figure 1.1), planar isotopies and* **self-crossing changes**, *i.e., crossing changes of arcs from the same component (see Figure 1.2).*

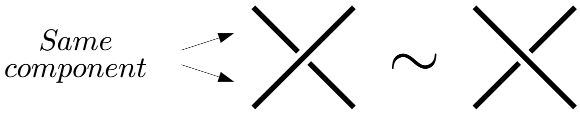


Figure 1.2: A self-crossing change.

Definition 1.1.6. *A disk T smoothly embedded in the interior of the* 3*-ball is called a* **clasper** for a tangle θ *if it satisfies the following three conditions:*

- *T is the embedding of a connected thickened uni-trivalent tree with a cyclic order at each trivalent vertex. Thickened univalent vertices are called* **leaves**, *and thickened trivalent vertices,* **nodes**.

- *θ intersects T transversely, and the intersection points are in the interior of the leaves of T.*

- *Each leaf intersects θ in at least one point.*

It should be noted that our definition differs from that of [Hab00b]; claspers as defined here are referred to in K. Habiro's terminology as strict tree-claspers.

Diagrammatically, a clasper is represented by a uni-trivalent graph corresponding to the one to be thickened. The trivalent vertices are thickened according to Figure 1.3. On the univalent vertices we specify how the corresponding leaves intersect θ, and we also indicate how the edges are twisted using markers called **half-twists** (see Figure 1.3).

Definition 1.1.7. *Let T be a clasper for a tangle θ. We define the* **degree** *of T, denoted by* $\deg(T)$, *as its number of nodes plus one, or equivalently, its number of leaves minus one. The* **support** *of T, denoted by* $\mathrm{supp}(T)$, *is defined to be the set of components of θ that intersect T.*

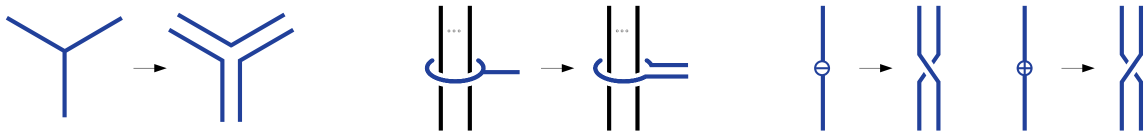


Figure 1.3: Local diagrammatic representation of claspers.

Definition 1.1.8. *A clasper T for a tangle θ is said to be* **simple** *if every leaf of T intersects θ exactly once. A leaf of a simple clasper intersecting the l-th component is called an l-*leaf*.*

Definition 1.1.9. *We say that a simple clasper T for a tangle θ has* **repeats** *if it intersects a component of θ in at least two points.*

Given a disjoint union of claspers F for a tangle θ, there is a procedure called *surgery* detailed in [Hab00b] to construct a new tangle, denoted θ^F. We illustrate on the left-hand side of Figure 1.4 the effect of a surgery on a clasper of degree one. Now if F contains some claspers with degree higher or equal than one, we first apply the rule shown on the right-hand side of Figure 1.4, at each trivalent vertex: this breaks up F into a disjoint union of degree one claspers, on which we can perform surgery.

Figure 1.4: Rules of clasper surgery.

Note that clasper surgery commutes with ambient isotopy. More precisely, for i an ambient isotopy and F a disjoint union of claspers for a tangle θ we have that $i(\theta^F) = (i(\theta))^{i(F)}$. This is an elementary example of *clasper calculus*, which refers to the set of operations on unions of tangles with some claspers, that allow to deform one into another with isotopic surgery result. These operations are developed in [Hab00b], and we give in the next section the analogous calculus up to link-homotopy.

1.1.2 Clasper calculus up to link-homotopy

In the whole section, T and S denote simple claspers for a given tangle θ. We use the notation $T \sim S$, and say that T and S are *link-homotopic* when the surgery results θ^T and θ^S are so. For example, if i is an ambient isotopy that fixes θ, then $T \sim i(T)$. Moreover, if θ^T is link-homotopic to θ, we say that T *vanishes up to link-homotopy* and we denote $T \sim \varnothing$.

We begin by recalling a fundamental lemma from [FY09]; more precisely, the next result is the case $k = 1$ of [FY09, Lemma 1.2], where self C_1-equivalence corresponds to link-homotopy.

Lemma 1.1.10. *[FY09, Lemma 1.2] If T has repeats, then T vanishes up to link-homotopy.*

It is well known to the experts that combining Lemma 1.1.10 with the proofs of K. Habiro's technical results on clasper calculus [Hab00b], yields the following *link-homotopy clasper calculus.*[1]

[1]Those moves are contained in [Yas09] and [MY12] together with [FY09].

Proposition 1.1.11. *[Hab00b, Proposition 3.23, 4.4, 4.5 and 4.6] We have the following link-homotopy equivalences (illustrated in Figure 1.5).*

(1) If S is a parallel copy of T which differs from T only by one half-twist (positive or negative), then $S \cup T \sim \varnothing$.

(2) If T and S have two adjacent leaves and if $T' \cup S'$ is obtained from $T \cup S$ by exchanging these leaves as depicted in (2) from Figure 1.5, then $T \cup S \sim T' \cup S' \cup \tilde{T}$, where $\tilde{T}$ is as shown in the figure.

(3) If T' is obtained from T by a crossing change with a strand of the tangle θ as depicted in (3) from Figure 1.5, then $T \sim T' \cup \tilde{T}$, where $\tilde{T}$ is as shown in the figure.

(4) If $T' \cup S'$ is obtained from $T \cup S$ by a crossing change between one edge of T and one of S as depicted in (4) from Figure 1.5, then $T \cup S \sim T' \cup S' \cup \tilde{T}$, where $\tilde{T}$ is as shown in the figure.

(5) If T' is obtained from T by a crossing change between two edges of T then $T \sim T'$.

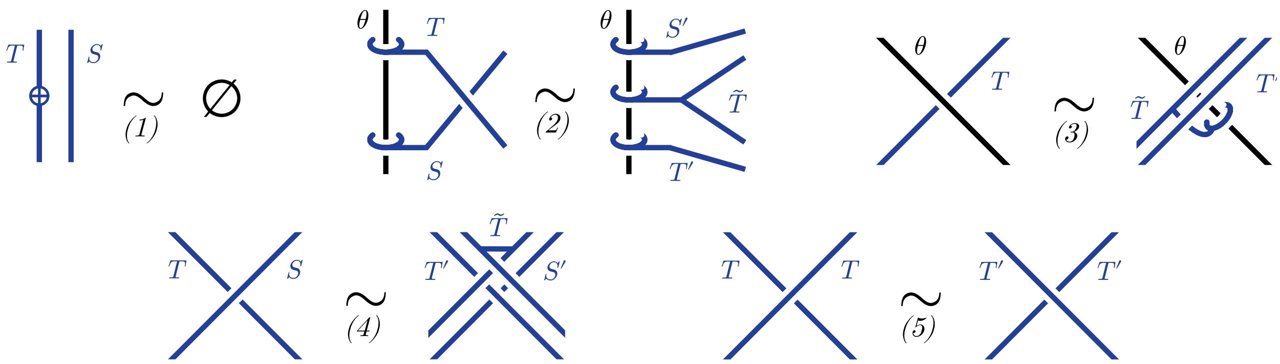


Figure 1.5: Basic clasper moves up to link-homotopy.

Idea of proof. The result of [Hab00b] used here are up to C_k*-equivalence*, that is, up to claspers of degree up to k. The key observation is that, by construction, all such higher degree claspers have same support as the initial ones, hence they are claspers with repeats. Lemma 1.1.10 then allows us to delete them up to link-homotopy. □

Remark 1.1.12. *Lemma 1.1.10 combined with Proposition 1.1.11 gives us some further results:*

- *First, statement (4) implies that if $|\mathrm{supp}(T) \cap \mathrm{supp}(S)| \geqslant 1$ then we can realize crossing changes between the edges of T and S.*

- *Moreover, if $|\mathrm{supp}(T) \cap \mathrm{supp}(S)| \geqslant 2$ thanks to statement (2) we can also exchange the leaves of T and S.*

- *Furthermore, statement (3) allows crossing changes between T and a component of θ in the support of T*

Indeed, in each case the clasper $\tilde{T}$ involved in the corresponding statement has repeats and can thus be deleted up to link-homotopy.

The next remark describes how to handle half-twists up to link-homotopy.

Remark 1.1.13. *We have the following link-homotopy equivalences (illustrated in Figure 1.6).*

(6) If T' is obtained from T by turning a positive half-twist into a negative one, then $T \sim T'$.

(7) If T' is obtained from T by moving a half-twist across a node then $T \sim T'$.

(8) If T and T' are identical outside a neighborhood of a node, and if in this neighborhood T and T' are as depicted in (8) from Figure 1.6, then $T \sim T'$.

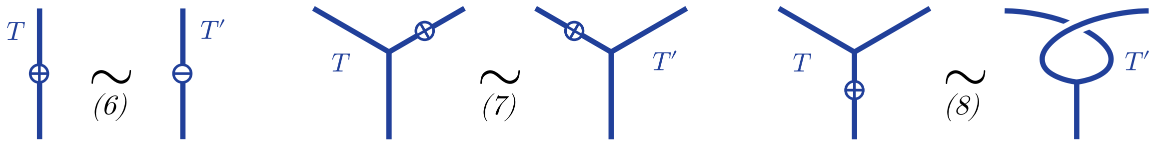


Figure 1.6: How to deal with half-twist up to link-homotopy.

Remark 1.1.14. *Remark 1.1.13 allows us to bring all the half-twists on a same edge and then cancel them pairwise. Therefore, we can consider only claspers with one or no half-twist.*

Proposition 1.1.11 together with Remark 1.1.13 give us most of the necessary tools to understand clasper calculus up to link-homotopy. The missing ingredient is the relation IHX which we give in the following proposition.

Proposition 1.1.15. *[CST07] Let T_I, T_H, T_X be three parallel copies of a given simple clasper that coincide everywhere outside a 3-ball, where they are as shown in Figure 1.7. Then $T_I \cup T_H \cup T_X \sim \varnothing$. We say that T_I, T_H and T_X* verify the IHX relation.

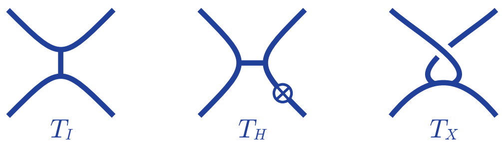


Figure 1.7: The IHX relation for claspers.

1.2 Reduced groups and commutators

In this document, the groups will be denoted multiplicatively, and $[a,b] := aba^{-1}b^{-1}$ will denote the commutator of two elements a,b.

Definition 1.2.1. *Let G be a group generated by $\{x_1, \ldots, x_n\}$. We define $J_G \lhd G$ to be the normal subgroup generated by elements of the form $[x_i, \lambda x_i \lambda^{-1}]$, for all $i \in \{1, \ldots, n\}$, and for all $\lambda \in G$. We call* **reduced quotient**, *the quotient G/J_G and we denote it by $\mathcal{R}G$.*

Remark 1.2.2. *This definition depends on the choice of the generators* $\{x_1, \dots, x_n\}$*. We will develop this point further in Chapter 5, when we discuss the notion of* fully reduced groups.

In what follows, we work essentially with the free group F_n on n generators $x_1, \dots, x_n$. The reduced quotient $\mathcal{R}F_n = F_n/J$ of the free group is called **reduced free group**, where $J := J_{F_n}$.

Definition 1.2.3. *Let* G *be a group and* $x_1, \cdots, x_n$ *elements in* G*. A* **commutator** in $x_1, \cdots, x_n$ of weight k $(k > 0)$ *can be defined recursively, as follows:*

- *The commutators of weight one are* $x_1, \dots, x_n$*.*
- *The commutators of weight* k *are words* $[C_1, C_2]$ *where* C_1*,* C_2 *are commutators verifying* $k = \mathrm{wg}(C_1) + \mathrm{wg}(C_2)$ *where* $\mathrm{wg}(C)$ *denotes the weight of* C*.*

Definition 1.2.4. *We denote* $\mathrm{Occ}_i(C) = r$ *and we say that* x_i occurs r times *in a commutator* C *if one of the following holds:*

- *If* $C = x_j$*, then* $r = 1$ *if* $i = j$ *and* $r = 0$ *if* $i \neq j$*.*
- *If* $C = [C_1, C_2]$*, then* $r = \mathrm{Occ}_i(C_1) + \mathrm{Occ}_i(C_2)$*.*

We say that a commutator C *has* **repeats** *if* $\mathrm{Occ}_i(C) > 1$ *for some* i*. We call* **support** *of the commutator* C*, the set of elements* x_i *(or by abuse just the set of indices* i*) such that* $\mathrm{Occ}_i(C) > 0$ *and we denote it* $\mathrm{supp}(C)$*.*

The following is a reformulation of Definition 1.2.1 that is used throughout the document.

Proposition 1.2.5. *[Lev88, Proposition 3] The subgroup* J *is generated by commutators in* $x_1, \dots, x_n$ *with repeats. Hence, these commutators are trivial in the reduced free group. The reduced quotient of a group* G *generated by* $x_1, \dots, x_n$ *is given by adding to* G*, the relations* $C = 1$ *for any commutator* C *with repeats in* $x_1, \dots, x_n$*.*

Corollary 1.2.6. *The subgroup* J *is generated by commutators in* $x_1, \dots, x_n$ *with repeats, subject to the condition* $\mathrm{wg}(C) \leqslant 2n$*. Equivalently, the reduced quotient of a group* G *generated by* $x_1, \dots, x_n$ *admits a finite set of relations given by* $C = 1$ *for any commutator* C *with repeats in* $x_1, \dots, x_n$*, satisfying the condition* $\mathrm{wg}(C) \leqslant 2n$*.* ·

Proof. Let us first observe that for any commutator $C = [C_1, C_2]$ satisfying $\mathrm{wg}(C) > 2n$, there exists (at least one) $i \in 1,2$ such that $\mathrm{wg}(C_i) > n$ and $\mathrm{wg}(C_i) < \mathrm{wg}(C)$. Also, note that any commutator $C = [C_1, C_2]$ belongs to the normal subgroups generated by C_1 and C_2. Thus, by iterating these two results, we demonstrate that any commutator of weight strictly greater than $2n$, and therefore with repetitions, is generated by a commutator C satisfying $n < \mathrm{wg}(C) \leqslant 2n$, and thus also has repetitions. This shows, in particular, that any commutator with repetitions is generated by commutators with repetitions of weight less than or equal to $2n$ and completes the proof. □

The notion of *basic commutators* was first introduced in [Hal33] and was further studied in [LS01, Hal59, MKS04] to describe the lower central series of the free group. It was then naturally adapted in [Lev88] to the framework of the reduced free group. In the next definition, we describe a well-chosen family of commutators. This family will replace the *reduced basic commutators* from [Lev88] and will follow us throughout the whole document.

Definition 1.2.7. *Let us define the following family of commutators without repeats in* $\mathcal{R}F_n$*:*

$$\mathcal{F} = \{[i_1,\ \ldots,\ i_l] \mid i_1 < i_k,\ 2 \leqslant k \leqslant l\}_{l \leqslant n}\,.$$

Here, we use the notation $[i_1,i_2,\cdots,\ i_l] := [[\cdots[[x_{i_1},x_{i_2}],x_{i_3}],\cdots,x_{i_{l-1}}],x_{i_l}]$*. This is a finite set and we can thus choose an arbitrary order on it,* $\mathcal{F} = \{[\alpha_1],\ [\alpha_2],\ \ldots,\ [\alpha_m]\}$*. We say that an element* $\omega \in \mathcal{R}F_n$ *is* in **normal form** with respect to the order *if*

$$\omega = [\alpha_1]^{e_1}[\alpha_2]^{e_2}\cdots[\alpha_m]^{e_m}$$

for some integers $\{e_1,\ e_2,\ \ldots,\ e_m\}$*.*

Definition 1.2.8. *We can define the order given for two commutators* $[\alpha] = [i_1\cdots i_l]$ *and* $[\alpha'] = [i'_1\cdots i'_{l'}]$ *by* $[\alpha] \leqslant [\alpha']$ *if:*

- $\mathrm{wg}(\alpha) < \mathrm{wg}(\alpha')$*, or*
- $\mathrm{wg}(\alpha) = \mathrm{wg}(\alpha')$ *and* $i_1\ldots i_l <_{\mathrm{lex}} i'_1\ldots i'_l$*.*

Example 1.2.9. *With respect to this order, the normal form of an element* $\omega \in \mathcal{R}F_3 = \langle x_1,\ x_2,\ x_3\rangle$ *is given by 8 integers* $\{e_1,\ \ldots,\ e_8\}$ *as follows:*

$$\omega = [1]^{e_1}[2]^{e_2}[3]^{e_3}[12]^{e_4}[13]^{e_5}[23]^{e_6}[123]^{e_7}[132]^{e_8}.$$

The following theorem is a kind of reduced analogue of Hall's basis theorem [Hal59, Theorem 11.2.4]. It is to be compared with [Lev88, Proposition 6], where a different family of commutators is used, see Remark 1.2.12.

Theorem 1.2.10. *For any word* $\omega \in \mathcal{R}F_n$ *there exists a unique ordered set of integers* $\{e_1,\ \ldots,\ e_m\}$ *associated to the ordered family of commutators* $\mathcal{F} = \{[\alpha_1],\ [\alpha_2],\ \ldots,\ [\alpha_m]\}$ *such that*

$$\omega = [\alpha_1]^{e_1}[\alpha_2]^{e_2}\cdots[\alpha_m]^{e_m}.$$

Proof. Let us first express any commutator C as a product of commutators in $\mathcal{F}$ with the same weight as C. To do so, we use the following three relations in $\mathcal{R}F_n$.

(i) $[X,Y]^{-1} = [Y,X] = [X^{-1},Y] = [X,Y^{-1}]$ with X,Y commutators.

(ii) $[X,[Y,Z]] = [[X,Y],Z]\cdot[[X,Z],Y]^{-1}$ with X,Y,Z commutators.

(iii) $[UV,X] = [U,X][V,X]$ with U,V commutators such that $\mathrm{supp}(U) \cap \mathrm{supp}(V) \neq \varnothing$.

Relation (i) allows us to move the generator x_{i_1} with $i_1 = \min(\mathrm{supp}(C))$ at the desired position; we obtain $C = [\cdots[x_{i_1},C_1],\cdots,C_k]^{\pm1}$. Relations (i), (ii) and (iii) are used to decrease the weight of the commutator C_i in this expression. We start with $C_1 = [C'_1,C'_2]$ supposing its weight is bigger than one, and we get:

$$\begin{aligned}
C &= [\cdots[x_{i_1},[C'_1,C'_2]],\cdots,C_k]^{\pm1}\\
&= [\cdots[[x_{i_1},C'_1],C'_2]\cdot[[x_{i_1},C'_2],C'_1]^{-1},\cdots,C_k]^{\pm1}\\
&= [\cdots[[x_{i_1},C'_1],C'_2],\cdots,C_k]^{\pm1}[\cdots[[x_{i_1},C'_2],C'_1]^{-1},\cdots,C_k]^{\pm1}\\
&= [\cdots[[x_{i_1},C'_1],C'_2],\cdots,C_k]^{\pm1}[\cdots[[x_{i_1},C'_2],C'_1],\cdots,C_k]^{\mp1}.
\end{aligned}$$

Since $\mathrm{wg}(C'_1) < \mathrm{wg}(C)$ and $\mathrm{wg}(C'_2) < \mathrm{wg}(C)$ we know that by iterating this operation on the new terms we can rewrite C as a product of commutators of the form $[\cdots[x_{i_1},x_{i_2}],C_2],\cdots,C_k]$, having in particular the same weight as C. We finish by repeating the process on $C_2, \ldots, C_k$.

For any $\omega \in \mathcal{R}F_n$, we can now demonstrate the existence of a decomposition $\omega = \prod_{\alpha\in\mathcal{F}}[\alpha]^{e_\alpha}$. We begin by expressing w as a product of weight 1 commutators belonging to $\mathcal{F}$. This is possible because weight 1 commutators in $\mathcal{F}$ are precisely the generators $x_1, \ldots, x_n$ of $\mathcal{R}F_n$. Next, we rearrange these weight-one commutators according to the order given by the family $\mathcal{F}$. This is achieved up to commutators of weight strictly higher than one, as two commutators commute up to commutators of strictly higher weight. Using the argument given at the beginning of this proof, we may safely assume that these higher weigh commutators belong to $\mathcal{F}$. We then consider, among these new commutators, those of weight two and rearrange them according to the order in $\mathcal{F}$. Again, this introduces higher weight factors, which can also be assumed to be elements of $\mathcal{F}$. By iterating this procedure, we eventually obtain the desired decomposition. Indeed, the procedure terminates because any commutator of weight strictly bigger than n has repeats and is then trivial according to Proposition 1.2.5.

To prove the unicity of the decomposition, we work with the unit group U_n of the ring of power series in non-commuting variables $X_1, \ldots, X_n$. More precisely, we consider its quotient $\tilde{U}_n$ in which the monomials $X^\alpha = X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_n}$ vanish when they have repetition (i.e., $\alpha_i = \alpha_j$ for some $i \neq j$). The elements that we will consider in $\tilde{U}_n$ are of the form $1+Q$ with Q a sum of monomials of positive degree, and their inverses are given by $(1+Q)^{-1} = 1+\bar{Q}$ with $\bar{Q} = -Q + Q^2 - Q^3 + \cdots (-1)^n Q^n$. Now we can define the *reduced Magnus expansion* $\tilde{M}$. This is a homomorphism from the reduced free group $\mathcal{R}F_n$ to $\tilde{U}_n$, defined by $\tilde{M}(x_i) = 1 + X_i$. The following computation shows that $\tilde{M}$ respects the relations of the reduced free group, meaning that $\tilde{M}([x_i,\lambda x_i\lambda^{-1}]) = 1$ for any generator x_i and any λ in F_n:

$$\begin{aligned}
\tilde{M}(\lambda x_i\lambda^{-1})\tilde{M}(x_i) &= \Big(\tilde{M}(\lambda)(1+X_i)\tilde{M}(\lambda^{-1})\Big)(1+X_i) \\
&= 1 + X_i + \tilde{M}(\lambda)X_i\tilde{M}(\lambda^{-1}) \\
&= (1+X_i)\Big(\tilde{M}(\lambda)(1+X_i)\tilde{M}(\lambda^{-1})\Big) \\
&= \tilde{M}(x_i)\tilde{M}(\lambda x_i\lambda^{-1}).
\end{aligned}$$

An easy induction on the weight l of the commutator $[\alpha] \in \mathcal{F}$ defined in Definition 1.2.7 gives the following:

Claim 1.2.11. *For every $[\alpha] = [\alpha_1,\cdots,\alpha_l] \in \mathcal{F}$, $\tilde{M}([\alpha]) = 1 + X^\alpha + Q_l(X_{\alpha_1},\cdots,X_{\alpha_l})$ where Q_l is a sum of monomials of degree $l = wg([\alpha])$ not starting with X_{α_1}, and where each variable X_{α_i} for $i \in \{1, \ldots, l\}$ appears exactly once.*

Now, we take $\omega = \prod_{\alpha\in\mathcal{F}}[\alpha]^{e_\alpha} = \prod_{\alpha\in\mathcal{F}}[\alpha]^{e'_\alpha}$ two decompositions of an element $\omega \in \mathcal{R}F_n$. We prove by induction on the weight of $[\alpha]$ that $e_\alpha = e'_\alpha$ for any commutator $[\alpha] \in \mathcal{F}$. Suppose that $e_\alpha = e'_\alpha$ for any $[\alpha]$ of weight $< k$ and compare the coefficients of the monomial X^α in both $\tilde{M}(\prod_{\alpha\in\mathcal{F}}[\alpha]^{e_\alpha})$ and $\tilde{M}(\prod_{\alpha\in\mathcal{F}}[\alpha]^{e'_\alpha})$ for a fixed commutator $[\alpha]$ of degree k. According to Claim 1.2.11, commutators of weight $> k$ do not contribute to this coefficient and the only contributing weight k commutator is $[\alpha]$ itself with coefficient e_α (resp. e'_α). Commutators of weight $< k$ may also contribute to this coefficient, but the induction hypothesis ensures that the contribution is the same in both expressions. This proves that $e_\alpha = e'_\alpha$ for any $[\alpha]$ of weight k and concludes the proof. □

Remark 1.2.12. *Unlike Levine's proof of [Lev88, Proposition 6], this proof does not require M. Hall's basis theorem [Hal59, Theorem 11.2.4].*

Definition 1.2.13. *To the ordered set of commutators $\mathcal{F} = \{[\alpha_1], \ldots, [\alpha_m]\}$ in $\mathcal{R}F_n$ we associate a* $\mathbf{Z}$*-module $\mathcal{V}$ formally generated by $\{\alpha_1, \ldots, \alpha_m\}$. We also define the linearization map $\phi : \mathcal{R}F_n \to \mathcal{V}$ by:*

$$\phi(\omega) = e_1\alpha_1 + \cdots + e_m\alpha_m \qquad \text{where } [\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m} \text{ is the normal form of } \omega.$$

We keep calling 'commutators' the generators of $\mathcal{V}$ and we define the support and the weight of α to be those of $[\alpha]$.

We stress that the normal form and the linearization map ϕ both depend on the ordering on $\mathcal{F}$.

Lemma 1.2.14. *The* $\mathbf{Z}$*-module $\mathcal{V}$ is of rank,*

$$rk(\mathcal{V}) = \sum_{0 \leqslant l \leqslant k < n} \frac{k!}{l!}.$$

Moreover we can decompose $\mathcal{V}$ into a direct sum of submodules $\mathcal{V}_i$ generated by the commutators of weight i. Then we obtain that:

$$rk(\mathcal{V}_i) = \binom{n}{i}(i-1)!.$$

Proof. The first equality comes by counting the cardinality of $\mathcal{F}$. To do so, we first count the elements $[\alpha]$ with first term $\alpha_1 = k$. To choose $\alpha_2, \alpha_3, \ldots, \alpha_l$ with $0 \leqslant l < n-k$ we only have to respect the condition that $\alpha_1 < \alpha_i$. Thus they can be freely chosen in $\{k+1, \ldots, n\}$ and therefore:

$$rk(\mathcal{V}) = \sum_{k=1}^{n} \sum_{l=1}^{n-k+1} \frac{(n-k)!}{(n-k-l+1)!} = \sum_{k=0}^{n-1} \sum_{l=0}^{k} \frac{k!}{(k-l)!} = \sum_{k=0}^{n-1} \sum_{l=0}^{k} \frac{k!}{l!}.$$

For the second equality, we follow the same kind of reasoning, but this time $\alpha_1 = k$ must be chosen in $\{1, \ldots, n-i+1\}$, then we choose the $i-1$ last numbers $\alpha_2, \ldots, \alpha_i$ without restriction in $\{k+1, \ldots, n\}$. We obtain:

$$rk(\mathcal{V}_i) = \sum_{k=1}^{n-i+1} \frac{(n-k)!}{(n-k-i+1)!} = \sum_{k=i-1}^{n-1} \frac{k!}{(k-i+1)!} = \left(\sum_{k=i-1}^{n-1} \binom{k}{i-1} \right) (i-1)!,$$

and we conclude using the so-called Hockey-stick identity. □

Chapter 2

Braids up to link-homotopy

This chapter is dedicated to the study of *braids* up to link-homotopy. In the next section, we introduce the notion of *comb-claspers* for braids, that yields a normal form result up to link-homotopy. Then, in Section 2.2, we give a new presentation of the *homotopy braid group* inspired from that of Goldsmith [Gol74] with a more symmetric structure. Section 2.3 deals with a linear representation of the homotopy braid group, defined and studied using clasper calculus. Finally, in Section 2.4, we begin to tackle the torsion problem, to which we provide a partial answer, to be completed later in chapter 4.

2.1 Braids and comb-claspers

Let D be the unit disk with n fixed points $\{p_i\}_{i\leqslant n}$ on a diameter δ, and let I be the unit interval $[0,\,1]$. Set also $I_1,\ \ldots,\ I_n$, n copies of I, and $\bigsqcup_{i\leqslant n} I_i$ their disjoint union.

Definition 2.1.1. *An n-*component **braid** $\beta = (\beta_1,\ \ldots,\ \beta_n)$ *is a smooth proper embedding*

$$(\beta_1,\ \ldots,\ \beta_n) : \bigsqcup_{i\leqslant n} I_i \to D \times I$$

*such that, for some permutation of $\{1,\ \ldots,\ n\}$ associated to β, denoted $\pi(\beta)$, we have $\beta_i(0) = (p_i,\, 0)$ and $\beta_i(1) = (p_{\pi(\beta)(i)},\, 1)$ for any i. We also require the embedding to be monotonic, which means that $\beta_i(t) \in D \times \{t\}$ for any $t \in [0,\,1]$. We call (the image of) β_i the i-*th component *of β. We say that a braid β is* **pure** *if its associated permutation $\pi(\beta)$ is the identity.*

We emphasize that braids are oriented from top to bottom; in particular, the interval I is parametrized in an unconventional manner, runing from '0' at the top to '1' at the bottom.

The composition of braids consists in stacking the braids one below the other: it is defined as follows. Let β and β' be two braids. Then their composition $\beta\beta'$ is a braid defined by

$$\beta\beta'_i(t) = \begin{cases} h_0(\beta_i(2t)), & \text{for } t \in [0,\tfrac{1}{2}], \\ h_1(\beta'_{\pi(\beta)(i)}(2(t-\tfrac{1}{2})), & \text{for } t \in [\tfrac{1}{2},1], \end{cases}$$

with $i \in [\![1,n]\!]$, and where the maps h_0 and $h_1 : D \times I \mapsto D \times I$ are defined for $x \in D$ and $t \in I$ by

$$h_0(x,t) = (x, \frac{1}{2} + \frac{t}{2}), \text{ and } h_1(x,t) = (x, \frac{t}{2}).$$

See Figure 2.1 for illustration.

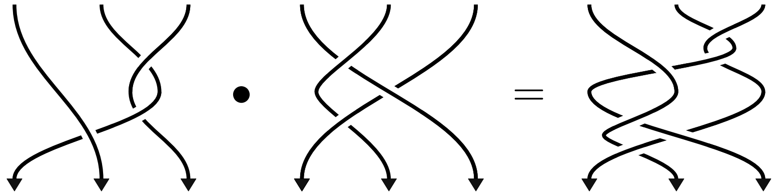


Figure 2.1: Example of composition of two braids.

Definition 2.1.2. *The set of braids up to ambient isotopy fixing the boundary, equipped with the stacking operation, forms a group. It is called the* **braid group** *and it is denoted by B_n. The set of pure braids up to isotopy forms a subgroup of B_n denoted by P_n and called the* **pure braid group**.

Definition 2.1.3. *The set of braids up to link-homotopy equipped with the stacking operation forms a group. It is called the* **homotopy braid group** *and it is denoted by hB_n. Elements of hB_n are called* homotopy braids. *The set of pure braids up to link-homotopy forms a subgroup of hB_n denoted by hP_n and called the* **pure homotopy braid group**.

Remark 2.1.4. *In [Art47], Artin raises the question of whether the notions of isotopy and link-homotopy of braids are different or identical. Goldsmith, in [Gol74], shows that the two notions are in fact different. As an illustration, we present Goldsmith's example of a braid in Figure 2.2, which is trivial up to link-homotopy but non-trivial up to isotopy.*

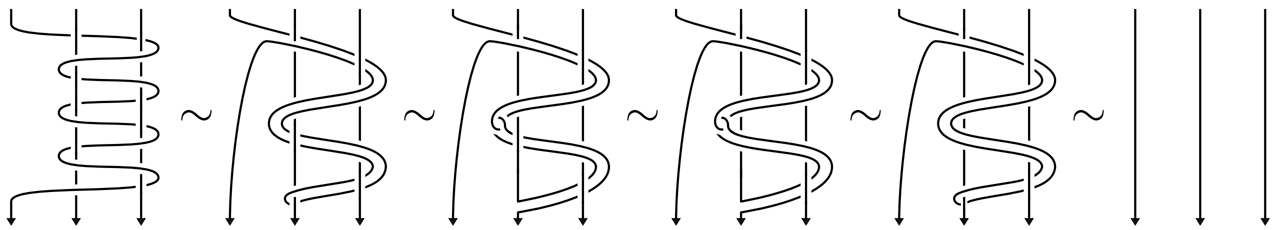


Figure 2.2: Example of a trivial braid up to link-homotopy, but non-trivial up to isotopy.

Remark 2.1.5. *Braids are tangles without closed components, and with boundary and monotonic conditions. But any (pure) tangle without closed components is link-homotopic to a (pure) braid (in the pure case, such tangles are called string-links in the literature). Thus, when regarding braids up to link-homotopy we can freely consider them as tangles, i.e., we can forget the monotonic condition. This is useful from the clasper point of view since clasper surgery does not respect this condition in general.*

We introduce next *comb-claspers* and their associated notation. Consider the usual representative $\mathbf{1}$ of the trivial n-component braid given by $\mathbf{1}_i = \{p_i\} \times I$ for $i \in \{1, \ldots, n\}$. Denote by $(D \times I)^+$ and $(D \times I)^-$ the two half-cylinders determined by the plane $\delta \times I$, where δ is the fixed diameter on D. In figures, we choose $(D \times I)^+$ to be above the plane of the projection.

Definition 2.1.6. *We call* **comb-clasper** *a simple clasper without repeats for the trivial braid such that:*

- *Every edge is in* $(D \times I)^+$.
- *The minimal path running from the smallest to the largest component of the support meets all nodes.*
- *At each node, the edge that does not belong to the minimal path leaves 'to the left' as locally depicted in Figure 2.3.*

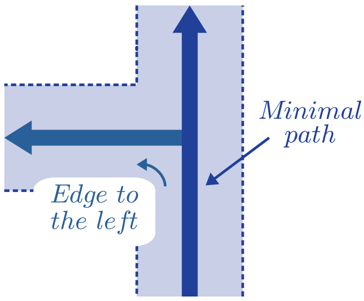


Figure 2.3: Local orientation at each node of a comb-clasper.

An example is given in Figure 2.4.

The second condition of Definition 2.1.6 implies that every node is connected (by an edge and a leaf) to a component of **1** that is not the smallest or the largest of the support. Using this fact, we can order the support of a comb-clasper: we start with the smallest component, then we order the components according to the order in which we meet them along the minimal path, and finally, we end with the largest one. For example, in Figure 2.4 the ordered support is $\{1, 2, 6, 4, 5, 8\}$.

Once the ordered support $\{i_1, i_2, \ldots, i_l\}$ is fixed, the only remaining indeterminacy in a comb-clasper is the embedding of the edges in $(D \times I)^+$. This depends on the relative position of the edges, and on the number of half-twists on each of them. However, up to link-homotopy the relative position of the edges is irrelevant (by move *(5)* from Proposition 1.1.11). Besides, by Remark 1.1.14, we can always suppose that a comb-clasper contains either one or no half-twist; moreover by Remark 1.1.13 we can freely assume that the potential half-twist is located on the edge connected to the i_l-th component. We can thus unambiguously (up to link-homotopy) denote by $(i_1,i_2,\cdots,i_l)$ the comb-clasper with such a half-twist and by $(i_1,i_2,\cdots,i_l)^{-1}$ the same clasper without any half-twist; we call them respectively **twisted** and **untwisted** comb-claspers. For example, the twisted comb-clasper (126458) is illustrated in Figure 2.4.

In what follows we blur the distinction between comb-claspers and the result of their surgery up to link-homotopy. From this point of view, a comb-clasper is a pure homotopy braid and the product $(\alpha)(\alpha')$ of two comb-claspers is the product $\mathbf{1}^{(\alpha)}\mathbf{1}^{(\alpha')}$. In particular, according to move *(1)* from Proposition 1.1.11, the inverse of a comb-clasper (α) is given by $(\alpha)^{-1}$.

Lemma 2.1.7. *Let T be a simple clasper of degree k for the trivial braid* $\mathbf{1}$*, then* $\mathbf{1}^T$ *is link-homotopic to a product of comb-claspers with degree greater than or equal to k.*

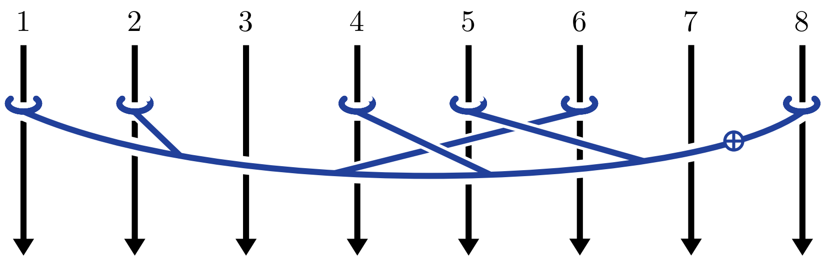


Figure 2.4: The twisted comb-clasper (126458).

Proof. First, we use isotopies and move *(3)* from Proposition 1.1.11 to turn T into a product of clasper with edges in $(D \times I)^+$. This step may create claspers of higher degree (corresponding to clasper $\tilde{T}$ in move *(3)*): in that case we also apply isotopies and move *(3)* on them until we get the desired product. Note that the procedure must stop. Indeed, move *(3)* always creates claspers of strictly higher degree, and when the degree is higher than the number of strands, the claspers have repetitions and are therefore trivial up to link-homotopy (Lemma 1.1.10). Then, by the IHX relation of Proposition 1.1.15, we can further assume that for each clasper, the minimal path running from the smallest to the largest component meets all its nodes. Finally, we apply move *(8)* from Remark 1.1.13 to satisfy the third condition of Definition 2.1.6 and obtain a product of comb-claspers. □

Definition 2.1.8. *We say that a pure homotopy braid $\beta \in hP_n$ given by a product of comb-claspers $\beta = (\alpha_1)^{\pm 1}(\alpha_2)^{\pm 1} \cdots (\alpha_m)^{\pm 1}$ is :*

- stacked *if $(\alpha_i) = (\alpha_j)$ for some $i \leqslant j$ implies that $(\alpha_i) = (\alpha_k)$ for any $i \leqslant k \leqslant j$,*
- reduced *if it contains no redundant pair, i.e., two consecutive factors are not the inverse of each other.*

If β is reduced and stacked, we can then rewrite $\beta = \prod (\alpha_i)^{\nu_i}$ for some integers ν_i and with $(\alpha_i) \neq (\alpha_j)$ for any $i \neq j$. Moreover, given an order on the set of twisted comb-claspers, we say that a reduced and stacked writing is a **normal form** of β for this order *if $(\alpha_i) \leqslant (\alpha_j)$ for all $i \leqslant j$.*

We stress that the notion of normal form is relative to a given order on the set of twisted comb-claspers. The following definition will be relevant for Chapter 3.

Definition 2.1.9. *Given two twisted comb-claspers $(\alpha) = (i_1 \cdots i_l)$ and $(\alpha') = (i'_1 \cdots i'_{l'})$ we can choose the order $(\alpha) \leqslant (\alpha')$ defined by:*

- $\max(\operatorname{supp}(\alpha)) < \max(\operatorname{supp}(\alpha'))$, *or*
- $\max(\operatorname{supp}(\alpha)) = \max(\operatorname{supp}(\alpha'))$ *and* $\deg(\alpha) < \deg(\alpha')$, *or*
- $\max(\operatorname{supp}(\alpha)) = \max(\operatorname{supp}(\alpha'))$ *and* $\deg(\alpha) = \deg(\alpha')$ *and* $i_1 \dots i_l <_{\text{lex}} i'_1 \dots i'_l$,

where $<_{\text{lex}}$ denotes the lexicographic order.

Example 2.1.10. *With respect to this order, the normal form of an element $\beta \in hP_4$ is given by 12 integers $\{\nu_{12}, \ldots, \nu_{1324}\}$ as follows:*

$$\beta = (12)^{\nu_{12}}(13)^{\nu_{13}}(23)^{\nu_{13}}(123)^{\nu_{123}}(14)^{\nu_{14}}(24)^{\nu_{24}}(34)^{\nu_{34}}(124)^{\nu_{124}}(134)^{\nu_{134}}(234)^{\nu_{234}}(1234)^{\nu_{1234}}(1324)^{\nu_{1324}}.$$

Theorem 2.1.11. *Any pure homotopy braid $\beta \in hP_n$ can be expressed in a normal form, for any order on the set of twisted comb-claspers.*

Proof. Note that the comb-clasper (ij) corresponds to the usual pure braid group generator $A_{ij} \in hP_n$ (see Figure 2.6). Thus it is clear that $\beta = \prod(\alpha)^{\pm 1}$ for some degree one comb-claspers $(\alpha)^{\pm 1}$.

Now we rearrange these degree one factors according to the chosen order by moves *(2)* and *(4)* from Proposition 1.1.11. This introduces new claspers of degree strictly higher than one, and by Lemma 2.1.7 we can freely assume that these are all comb-claspers. Next we consider, among these new comb-claspers, those of degree two and we rearrange them according to the order. Again this introduces higher degree factors, which can all be assumed to be comb-clasper according to Lemma 2.1.7. By iterating this procedure degree by degree, we eventually obtain the desired normal form. Indeed, the procedure terminates because claspers of degree higher or equal than n are trivial in hP_n by Lemma 1.1.10. □

Remark 2.1.12. *This result is to be compared with Theorem 4.3 of [Yas09], which uses a different notion of comb-clasper, ordered according to the clasper degree.*

2.2 Braid group presentations

In this section, we use the usual Artin braid generators σ_i for $i \in \{1, \ldots, n-1\}$ illustrated in Figure 2.5 and the usual pure braid generators $A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$ for $1 \leqslant i < j \leqslant n$, illustrated in Figure 2.6.

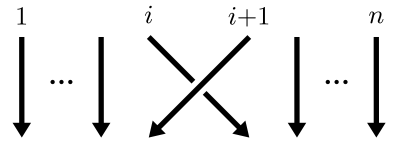


Figure 2.5: The Artin generator σ_i.

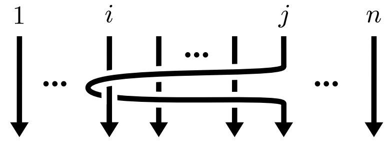


Figure 2.6: The pure braid generator A_{ij}.

We first recall the usual presentations of the braid group from [Art47] and the pure braid group from [Bir74].

Theorem 2.2.1. *A presentation*[1] *for the braid group is given by:*

$$B_n = \left\langle \sigma_i \;\middle|\; \begin{array}{ll} [\sigma_i,\sigma_j] = \mathbf{1} & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| \leqslant 1 \end{array} \right\rangle.$$

A presentation for the pure braid group is given by:

$$P_n = \left\langle A_{ij} \;\middle|\; \begin{array}{ll} [A_{rs},A_{ij}] = \mathbf{1} & \text{for } r<s<i<j \text{ or } r<i<j<s \\ [A_{rs},A_{rj}] = [A_{sj}^{-1},A_{rj}] & \text{for } r<s<j \\ [A_{rs},A_{sj}] = [A_{sj}^{-1},A_{rj}^{-1}] & \text{for } r<s<j \\ [A_{ri},A_{sj}] = [[A_{ij}^{-1},A_{rj}^{-1}],A_{sj}] & \text{for } r<s<i<j \end{array} \right\rangle.$$

[1]For the sake of compactness, here and in all presentations of the chapter, generators are indexed as above. That is, generators σ_i are indexed by integers $i \in \{1, \ldots, n-1\}$, and generators A_{ij} by pairs of integers $1 \leqslant i < j \leqslant n-1\}$.

The following theorem is based on the result of [Gol74].

Proposition 2.2.2. *Let $J \lhd B_n$ denote the normal subgroup generated by all elements of the form $[A_{ij},\lambda A_{ij}\lambda^{-1}]$ where λ belongs to P_n. We obtain the homotopy braid group hB_n as the quotient:*

$$hB_n = B_n/J.$$

This induces the following presentation[2] for hB_n:

$$hB_n = \left\langle \sigma_i \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| \leqslant 1 \\ [\sigma_i,\sigma_j] = \mathbf{1} & \text{if } |i-j| > 1 \\ \left[A_{ij},\lambda A_{ij}\lambda^{-1}\right] = \mathbf{1} & \text{for } i<j \text{ and } \lambda \in hP_n \end{array} \right\rangle.$$

Proof. In [Gol74], the homotopy braid group hB_n appears as the quotient B_n/J', where $J' \lhd B_n$ is the normal subgroup generated by elements of the form $[A_{ij},\lambda A_{ij}\lambda^{-1}]$ where λ belongs to the normal subgroup generated by $\{A_{1,j}, \ldots, A_{j-1,j}\}$. Our result relies on the observation that $J = J'$. Obviously $J' \subset J$ thus we only need to show that $J \subset J'$. This is equivalent to showing that for any $\lambda \in P_n$, the pure braid A_{ij} and $\lambda A_{ij}\lambda^{-1}$ commute up to link-homotopy. Let us remind that A_{ij} is the surgery result $\mathbf{1}^{(ij)}$ of the comb-clasper (ij). Take Λ a given representative of λ, and consider an ambient isotopy ι sending $\Lambda\mathbf{1}\Lambda^{-1}$ to the trivial braid $\mathbf{1}$. Now, consider the comb-clasper (ij) as a clasper for the braid $\Lambda\mathbf{1}\Lambda^{-1}$ and denote it by $\Lambda(ij)\Lambda^{-1}$. Apply ι to the braid $\Lambda\mathbf{1}\Lambda^{-1}$ together with the clasper $\Lambda(ij)\Lambda^{-1}$. This isotopy sends $\Lambda(ij)\Lambda^{-1}$ to a clasper for the trivial braid, denoted C, whose surgery result is the conjugate $\lambda A_{ij}\lambda^{-1}$. Since ambient isotopies preserve the support, it is clear that $\text{supp}(C) = \text{supp}(\Lambda(ij)\Lambda^{-1}) = \{i,j\}$. Hence, according to Remark 1.1.12, we have $(ij)C \sim C(ij)$, and the result is proved. □

Remark 2.2.3. *The presentation provided in Proposition 2.2.2 is not a finite presentation due to the infinite set of reduced-type relations $\left[A_{ij},\lambda A_{ij}\lambda^{-1}\right] = \mathbf{1}$. However, by using the characterization in terms of repeated commutators, as seen in Proposition 1.2.5, we can use Corollary 1.2.6 how to reduce it to a finite set of relations.*

Remark 2.2.4. *This proposition can also be demonstrated purely algebraically. It was the subject of the master's thesis of I. Mazzotti, which I co-supervised in Caen. The proof is much more technical and it is based on commutator calculus [MK99] and braid group presentation results [Gol74, Min15].*

In order to obtain a similar result for the pure homotopy braid group we need the following.

Lemma 2.2.5. *The subgroup $J \lhd B_n$ normally generated in B_n by elements of the form $[A_{ij},\lambda A_{ij}\lambda^{-1}]$ for $\lambda \in P_n$, seen as a subgroup of P_n, coincides with the normal subgroup of P_n generated by elements of the form $[A_{ij},\lambda A_{ij}\lambda^{-1}]$ for $\lambda \in P_n$.*

Proof. For $k \in \{1, \ldots, n-1\}$, $1 \leqslant i < j \leqslant n$ and $\lambda \in P_n$ we compute:

$$\sigma_k[A_{ij},\lambda A_{ij}\lambda^{-1}]\sigma_k^{-1} = \begin{cases} [A_{i+1j},\lambda_1 A_{i+1j}\lambda_1^{-1}] & \text{if } i=k \text{ and } j \neq k+1 \\ [A_{i+1j},\lambda_2 A_{i+1j}\lambda_2^{-1}] & \text{if } j=k \\ A_{kk+1}[A_{i-1j},\lambda_3 A_{i-1j}\lambda_3^{-1}]A_{kk+1}^{-1} & \text{if } i=k+1 \\ A_{kk+1}[A_{ij-1},\lambda_4 A_{ij-1}\lambda_4^{-1}]A_{kk+1}^{-1} & \text{if } i \neq k \text{ and } j=k+1 \\ [A_{ij},\lambda_5 A_{ij}\lambda_5^{-1}] & \text{otherwise,} \end{cases}$$

[2] By the notation $\lambda \in P_n$ here we mean that λ is a pure homotopy braid, i.e., a word in the pure homotopy braid generators $\{A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}\}$ and their inverses.

with $\lambda_i \in P_n$ for $i \in \{1,2,3,4,5\}$. Therefore, the conjugates $\sigma_k[A_{ij},\lambda A_{ij}\lambda^{-1}]\sigma_k^{-1}$ are always conjugates of $[A_{i'j'},\lambda' A_{i'j'}(\lambda')^{-1}]$ in P_n for some $1 \leqslant i' < j' \leqslant n$ and some $\lambda' \in P_n$, and the proof is complete. □

Corollary 2.2.6. *Let $J \lhd P_n$ be the normal subgroup generated by elements of the form $[A_{ij},\lambda A_{ij}\lambda^{-1}]$ for any $\lambda \in P_n$. We obtain the pure homotopy braid group hP_n as the reduced quotient relative to the generative system $\{A_{ij} \mid i < j\}$ of the pure braid group:*

$$hP_n = P_n/J = \mathcal{R}P_n.$$

This induces the following presentation for hP_n:

$$hP_n = \left\langle A_{ij} \;\middle|\; \begin{array}{ll} [A_{rs},A_{ij}] = 1 & \text{for } r<s<i<j \text{ or } r<i<j<s \\ [A_{rs},A_{rj}] = [A_{rj},A_{sj}] = [A_{sj},A_{rs}] & \text{for } r<s<j \\ [A_{ri},A_{sj}] = [[A_{ij},A_{rj}],A_{sj}] & \text{for } r<s<i<j \\ \left[A_{ij},\lambda A_{ij}\lambda^{-1}\right] = \mathbf{1} & \text{for } i<j \text{ and } \lambda \in hP_n \end{array} \right\rangle.$$

Proof. The first half of the statement is a direct consequence of Proposition 2.2.2 and Lemma 2.2.5. The presentation is obtained from that of Theorem 2.2.1, using the relation $[A_{rs},A_{ij}^{-1}] = [A_{rs},A_{ij}]^{-1}$ which holds in $\mathcal{R}P_n$. □

Remark 2.2.7. *Once again, in Corollary 2.2.6, we provide an infinite presentation of the pure homotopy braid group seen as a reduced quotient. However, using Corollary 1.2.6, we can simplify this type of presentation to obtain a finite one.*

We next recall two classical representations of braid groups.

Definition 2.2.8. *We call* **Artin representation** *the homomorphism $\rho : B_n \to Aut(F_n)$ defined as follows:*

$$\rho(\sigma_i) : \begin{cases} x_i & \mapsto \quad x_{i+1}, \\ x_{i+1} & \mapsto \quad x_{i+1}x_ix_{i+1}^{-1}, \\ x_k & \mapsto \quad x_k \qquad \text{if } k \notin \{i,\ i+1\}. \end{cases}$$

Similarly, the homomorphism $\rho_h : hB_n \to Aut(\mathcal{R}F_n)$ defined by the same expressions is called the **homotopy Artin representation***.*

As the name suggests, ρ was introduced by Artin in [Art47], where its faithfulness is also shown. As for the link-homotopic version ρ_h, it is proved in [HL90] that its restriction to the pure homotopy braid group is faithful. Furthermore, for any braid $\beta \in hB_n$ and any generator $x_i \in \mathcal{R}F_n$, the image $\rho_h(\beta)(x_i)$ is a conjugate of $x_{\pi^{-1}(\beta)(i)}$. In particular, the kernel of ρ_h must belong to the pure homotopy braid group. The homotopy Artin representation ρ_h is therefore clearly faithful.

2.3 A linear faithful representation of the homotopy braid group

This section is devoted to the definition and study of a faithful linear representation of the homotopy braid group. We first define it algebraically, then give a procedure based on clasper calculus to compute it explicitly. Finally, we show its injectivity and use it to prove the uniqueness of the normal form in the homotopy braid group.

2.3.1 Algebraic definition

Let $GL(\mathcal{V})$ be the general linear group of the $\mathbf{Z}$-module $\mathcal{V}$ introduced in Definition 1.2.13. In order to define our linear representation $\gamma : hB_n \to GL(\mathcal{V})$, we state the following preparatory lemma. Let us denote by N_j the subgroup normally generated by x_j in $\mathcal{R}F_n$ for $j \in \{1, \ldots, n\}$; note that N_j is an abelian group.

Lemma 2.3.1. *Let $\beta \in hB_n$ be a homotopy braid and $C \in N_j$ a commutator in $\mathcal{R}F_n$. If the product $[\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m}$ is the normal form of $\rho_h(\beta)(C)$ (associated to a given order), then we have that $e_i = 0$ if $[\alpha_i] \notin N_{\pi^{-1}(\beta)(j)}$. Here $\pi^{-1}(\beta)(j)$ is the image of j by the permutation induced by β^{-1}.*

In other words, in the image of $C \in N_j$ by $\rho_h(\beta)$, the letter $x_{\pi^{-1}(\beta)(j)}$ occurs in each factor of the normal form.

Proof. Note first that any element of N_j is sent by $\rho_h(\beta)$ to an element of $N_{\pi^{-1}(\beta)(j)}$. This is clear for the Artin generators σ_i, and so is it for any braid β. Next, for a given integer $k \in \{1, \cdots, n\}$, consider the endomorphism of $\mathcal{R}F_n$ defined by $x_i \mapsto 1$, if $i = k$ and $x_i \mapsto x_i$, otherwise. This endomorphism sends a commutator to 1 if it belongs to N_k and to itself otherwise. In addition, it sends the normal form of any $\omega \in N_k$ to the normal form of 1. So by unicity of the normal form in $\mathcal{R}F_n$ (Theorem 2.3.12), for any $\omega \in N_k$, the normal form $\omega = [\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m}$ contains only commutators in N_k, i.e., $e_i = 0$ if $[\alpha_i] \notin N_k$. □

Recall from Definition 1.2.13 the linearization map $\phi : \mathcal{R}F_n \to \mathcal{V}$. Recall also from definition 1.2.7 the family $\mathcal{F}$ of (basic) commutator in $\mathcal{R}F_n$.

Proposition 2.3.2. *The map*

$$\gamma : hB_n \to GL(\mathcal{V})$$

defined for $\beta \in hB_n$ and $[\alpha] \in \mathcal{F}$ by $\gamma(\beta)(\alpha) = \phi \circ \rho_h(\beta)([\alpha])$ is a well-defined homomorphism. Moreover, γ does not depend on the chosen order on $\mathcal{F}$.

Proof. Since ϕ is not a homomorphism in general, it is not clear that γ is a representation. Yet we do have that $\gamma(\beta\beta') = \gamma(\beta)\gamma(\beta')$ for any two homotopy braids β and β', which is shown as follows. Let $[\alpha]$ be a commutator in $\mathcal{F}$ and α its corresponding commutator in $\mathcal{V}$. We choose some $j \in \mathrm{supp}([\alpha])$ so that $[\alpha]$ is in N_j. Set $\gamma(\beta')(\alpha) = \sum_i e_i \alpha_i$ for some commutators $\alpha_i \in \mathcal{V}$ associated to the commutators $[\alpha_i] \in \mathcal{F}$ and some integers e_i. Then we have

$$\gamma(\beta\beta')(\alpha) = \phi \circ \rho_h(\beta)\rho_h(\beta')([\alpha]) = \phi \circ \rho_h(\beta)\Big(\prod_i [\alpha_i]^{e_i}\Big) = \phi\Big(\prod_i \rho_h(\beta)([\alpha_i])^{e_i}\Big).$$

Now, using Lemma 2.3.1 we know that $[\alpha_i]$ is in $N_{\pi^{-1}(\beta')(j)}$ for any i. Moreover, Lemma 2.3.1 implies that any commutator in the normal form of $\rho_h(\beta)([\alpha_i])$ is in the abelian group $N_{\pi^{-1}(\beta\beta')(j)}$ for any i. But note that for $C_1, \ldots, C_k$ a collection of commutators in $\mathcal{F}$ such that $[C_i, C_j] = 1$ for any i, j, we have that $\phi(C_1 \cdots C_k) = \phi(C_1) + \cdots + \phi(C_k)$. Hence ϕ behaves like a homomorphism on the product $\prod_i \rho_h(\beta)([\alpha_i])^{e_i}$, and finally,

$$\phi\Big(\prod_i \rho_h(\beta)([\alpha_i])^{e_i}\Big) = \sum_i e_i \phi\Big(\rho_h(\beta)([\alpha_i])\Big) = \sum_i e_i \gamma(\beta)(\alpha_i) = \gamma(\beta)\Big(\sum_i e_i(\alpha_i)\Big) = \gamma(\beta)\gamma(\beta')(\alpha).$$

This shows that γ is a well-defined homomorphism.

To prove the independence on the chosen order on $\mathcal{F}$ we use Lemma 2.3.1 again. For any $\beta \in hB_n$ and any $[\alpha] \in \mathcal{F}$, all the commutators in the normal form of $\rho_h(\beta)([\alpha])$ commute with each other. In particular, if we set two orderings $\{[\alpha_1], \ldots, [\alpha_m]\}$ and $\{[\alpha_{\sigma(1)}], \ldots, [\alpha_{\sigma(m)}]\}$ on $\mathcal{F}$ then the two associated normal forms

$$\rho_h(\beta)([\alpha]) = [\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m} = [\alpha_{\sigma(1)}]^{e'_{\sigma(1)}} \cdots [\alpha_{\sigma(m)}]^{e'_{\sigma(m)}}$$

satisfy $e_i = e'_i$ for any i and therefore $\phi \circ \rho_h = \phi' \circ \rho_h$ for the two linearization maps ϕ and ϕ' associated to the orderings. □

Remark 2.3.3. *The homomorphism γ is in fact injective. Since ϕ is clearly injective, this can be shown using the injectivity of ρ_h. However, we will give below another proof of this result in Theorem 2.3.11 using clasper calculus, which in turn reproves the injectivity of ρ_h. Furthermore, our approach by clasper calculus allows explicit computations of the representation, as shown in the next section.*

2.3.2 Clasper interpretation

We first give a topological interpretation of the Artin (resp. homotopy Artin), representation. We can see the free group F_n (resp. reduced free group $\mathcal{R}F_n$) on which B_n (resp. hB_n) acts, as the fundamental group (resp. the reduced fundamental group) of the complement of the n-component trivial braid. Therefore, an element of F_n (resp. $\mathcal{R}F_n$) can also be seen as the homotopy (resp. the *reduced homotopy*[3]) class of an $(n+1)$-th component in this complement. On the diagram, we place this new strand to the right of the braid and we label it by '∞'. Thus, the generators x_i of F_n (resp $\mathcal{R}F_n$) are given by the pure braids $A_{i\infty}$ shown in Figure 2.7, which can be reinterpreted with the comb-claspers (i,∞) depicted in the same figure. There and in subsequent figures, we simply represent with a circled '∞' the leaf intersecting the ∞-th component. In this context, the automorphism $\rho(\beta)$

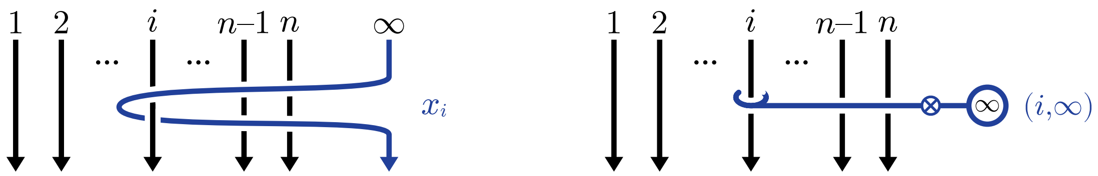


Figure 2.7: Pure braid and clasper interpretations of the generator x_i.

(resp. $\rho_h(\beta)$) associated to an element β in B_n (resp. hB_n) is given on a generator $x_i \in F_n$ (resp. $\mathcal{R}F_n$) by considering the conjugation $\beta\mathbf{1}^{(i,\infty)}\beta^{-1}$ illustrated in Figure 2.8. Then, we apply an isotopy, transforming $\beta\mathbf{1}\beta^{-1}$ into $\mathbf{1}$, as in the proof of 2.2.2. By doing so the clasper (i,∞) is deformed into a new clasper which we are able to reinterpret as an element of F_n or $\mathcal{R}F_n$.

In this way, we obtain an explicit procedure to compute our representation γ from Proposition 2.3.2 using clasper calculus, as follows. Given $\beta \in hB_n$ and $\alpha \in \mathcal{V}$, the computation of $\gamma(\beta)(\alpha)$ goes in 3 steps:

Step 1 Consider the conjugate of the comb-clasper (α,∞) by the braid β (see Figure 2.8).

[3]Here by *reduced homotopy class*, we mean the image in the reduced quotient of the homotopy class of an element.

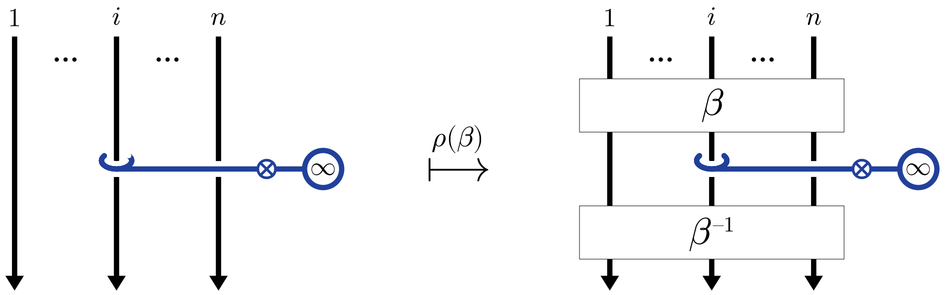


Figure 2.8: Clasper interpretation of the Artin representation.

Step 2 Use clasper calculus to re-express this conjugate as an ordered union of comb-claspers with ∞ in their support (the order comes from the order on $\mathcal{F}$).

Step 3 The number of parallel copies of a given comb-clasper in this product is the coefficient of the associated commutator in $\gamma(\beta)(\alpha)$.

Explicit examples of computations using this procedure are given in the proof of Theorem 2.3.5 below.

We note that we have a nice correspondence between the family $\mathcal{F}$, of commutators, and the comb-claspers having ∞ in their support, by the following proposition.

Proposition 2.3.4. *Let* $(\alpha) = (i_1 \cdots i_{n-1}\infty)$ *and* $(\alpha') = (i_1 \cdots i_{n-1} i_n \infty)$ *be two comb-claspers. Then we have the relation:*

$$(\alpha') \sim [(\alpha),(i_n\infty)] = (\alpha) \cdot (i_n\infty) \cdot (\alpha)^{-1} \cdot (i_n\infty)^{-1}.$$

For example in Figure 2.9 we illustrate the equivalence $(1254\infty) \sim [(125\infty),(4\infty)]$.

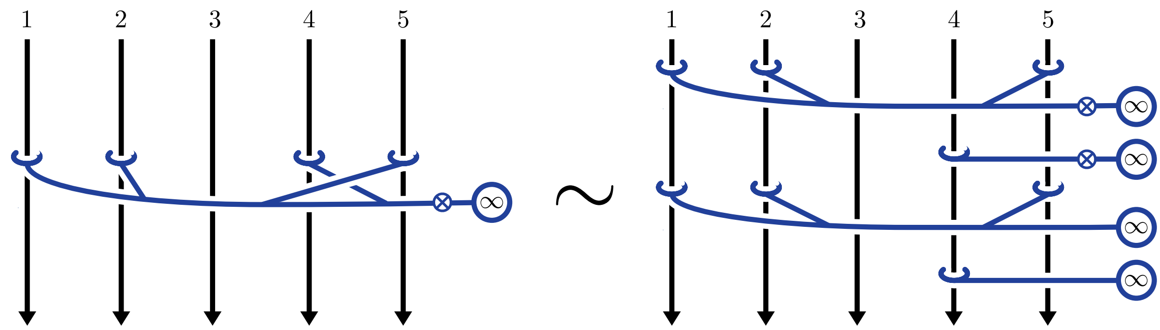


Figure 2.9: The comb-clasper (1254∞) is link-homotopic to the commutator $[(125\infty),(4\infty)]$.

Proof. Consider the product of comb-claspers $\alpha \cdot (i_n\infty) \cdot \alpha^{-1} \cdot (i_n\infty)^{-1}$ (as for example on the right-hand side of Figure 2.9). First, we use move (2) from Proposition 1.1.11 to exchange the ∞-th leaves of $(i_n\infty)$ and $(\alpha)^{-1}$; this move creates an extra comb-clasper, which is exactly (α'). Now by Remark 1.1.12 we can freely move (α') and finish exchanging the edges of $(\alpha_n\infty)$ and $(\alpha)^{-1}$, thus obtaining the product $(\alpha) \cdot (\alpha)^{-1} \cdot (\alpha') \cdot (i_n\infty) \cdot (i_n\infty)^{-1} \sim (\alpha')$. □

In practice, by iterating this proposition, we obtain a correspondence between the commutators $[\alpha] \in \mathcal{F}$ (or $\alpha \in \mathcal{V}$) and the comb-claspers (α,∞). For example the equivalence $(1254\infty) \sim [[[(1\infty),(2\infty)],(5\infty)],(4\infty)]$ corresponds to $[1254] = [[[x_1,x_2],x_5],x_4]$ in $\mathcal{R}F_n$.

2.3.3 Explicit computations

We now apply the 3-steps procedure of Section 2.3.2, to compute our representation γ for each generator σ_i of hB_n and each commutator in $\mathcal{V}$. In general, the image of the commutator $(i_1,i_2,\cdots,i_l) := \phi([i_1,i_2,\cdots,i_l]) \in \mathcal{V}$ by the map $\gamma(\sigma_i)$ depends on the position of the indices i and $i+1$ in the sequence $i_1,\ i_2,\ \ldots,\ i_l$, as stated in Theorem 2.3.5 below. Note that a program in *Python* that computes explicitly the representation γ is available on [Gra22].

Theorem 2.3.5. *For suitable sequences $I,\ J,\ K$ in $\{1,\ \ldots,\ n\}\backslash\{i,\ i+1\}$, $I \neq \varnothing$, we have:*

$$\gamma(\sigma_i) : \begin{cases} (I) & \mapsto \quad (I) & (a)\\ (J,i,K) & \mapsto \quad (J,i+1,K) & (b)\\ (i+1,K) & \mapsto \quad (i,K) + (i,i+1,K) & (c)\\ (I,i+1,K) & \mapsto \quad (I,i,K) + (I,i,i+1,K) - (I,i+1,i,K) & (d)\\ (I,i,J,i+1,K) & \mapsto \quad (I,i+1,J,i,K) & (e)\\ (I,i+1,J,i,K) & \mapsto \quad (I,i,J,i+1,K) & (f)\\ (i,J,i+1,K) & \mapsto \quad \sum_{J'\subseteq J}(-1)^{|J'|+1}(i,\overline{J'},i+1,J\backslash J',K) & (g) \end{cases}$$

where in (g), the sum is over all (possibly empty) subsequences J' of J, and $\overline{J'}$ denotes the sequence obtained from J' by reversing the order of its elements, see Example 2.3.6.

Example 2.3.6. *If $J = (j_1,\ j_2,\ j_3)$ and $K = \varnothing$ in (g), then $\gamma(\sigma_i)$ maps $(i,J,i+1)$ to :*

$$-(i,i+1,j_1,j_2,j_3) + (i,j_1,i+1,j_2,j_3) + (i,j_2,i+1,j_1,j_3) + (i,j_3,i+1,j_1,j_2)$$

$$-(i,j_2,j_1,i+1,j_3) - (i,j_3,j_1,i+1,j_2) - (i,j_3,j_2,i+1,j_1) + (i,j_3,j_2,j_1,i+1).$$

The proof below explains how this follows from the IHX relations of Figure 2.14.

Proof of Theorem 2.3.5. Following the 3-steps procedure of Section 2.3.2, we consider the conjugate $\sigma_i(\alpha,\infty)\sigma_i^{-1}$ and apply clasper calculus to turn it into a union of comb-claspers.

For (a) it is clear that (I,∞) commutes with σ_i, passing over or next to it. The computation of (b) is given by a simple isotopy of the braid shown in Figure 2.10.

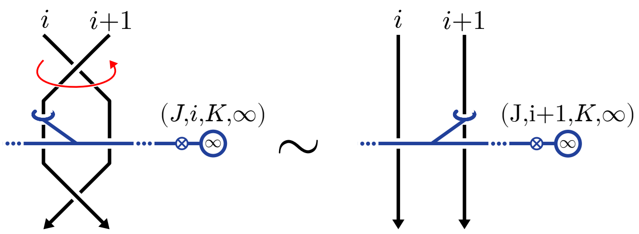


Figure 2.10: Computation of (b).

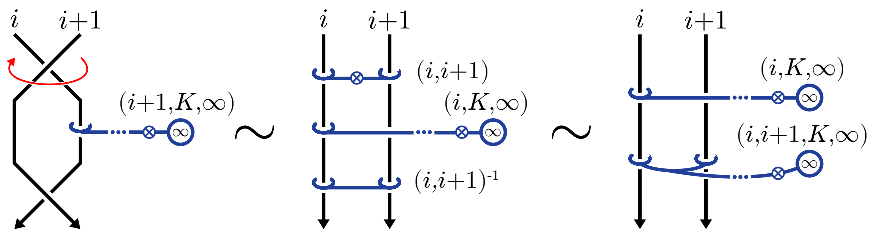


Figure 2.11: Computation of (c).

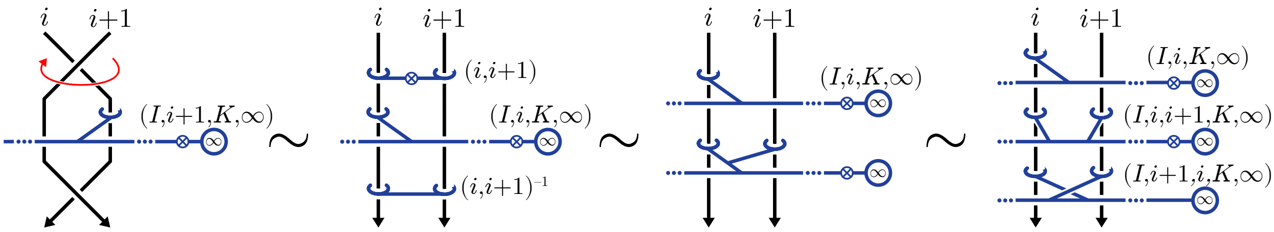


Figure 2.12: Computation of (d).

The proofs of (c) and (d) are similar and are given in Figures 2.11 and 2.12 respectively. There, the first equivalence is an isotopy, and the second one is given by move *(2)* from Proposition 1.1.11. For (d) there is a further step given by an IHX relation.

For (e) and (f) we apply the same isotopy as Figure 2.10 on components i and $i+1$, thus interchanging $(I,i,J,i+1,K)$ and $(I,i+1,J,i,K)$. Note that we also need a crossing change between the $(i+1)$-th component and a clasper edge, which is possible according to Remark 1.1.12.

Proving (g) is the last and hardest part and goes in two steps. The first step is illustrated in Figure 2.13: we proceed as before with an isotopy and a crossing change, then we use move *(8)* of Remark 1.1.13. This turns $\sigma_i(i,J,i+1,K,\infty)\sigma_i^{-1}$ into a new clasper which is not a comb-clasper.

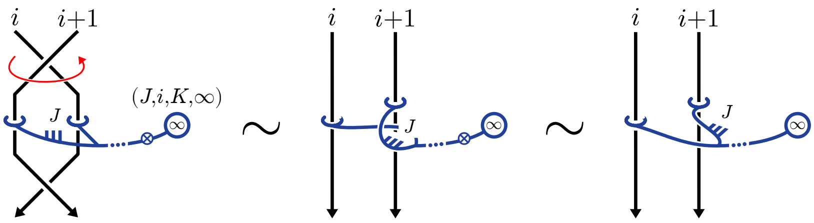


Figure 2.13: Turning $\sigma_i(i,J,i+1,K,\infty)\sigma_i^{-1}$ into a new clasper.

In the second step, we use IHX relations repeatedly to turn this new clasper into a product of comb-claspers. This is illustrated in Figure 2.14 where $J = (j_1,j_2,j_3)$. We conclude by simplifying the half-twists with Remark 1.1.14. □

Example 2.3.7. *We illustrate Theorem 2.3.5 by computing completely and explicitly the representation* γ *on the 3-component homotopy braid group* hB_3*. To do so, we set* (1), (2), (3), (12), (13), (23), (123) *and* (132) *to be the generators of* $\mathcal{V}$*, with the order of Definition 1.2.8, and we compute* γ

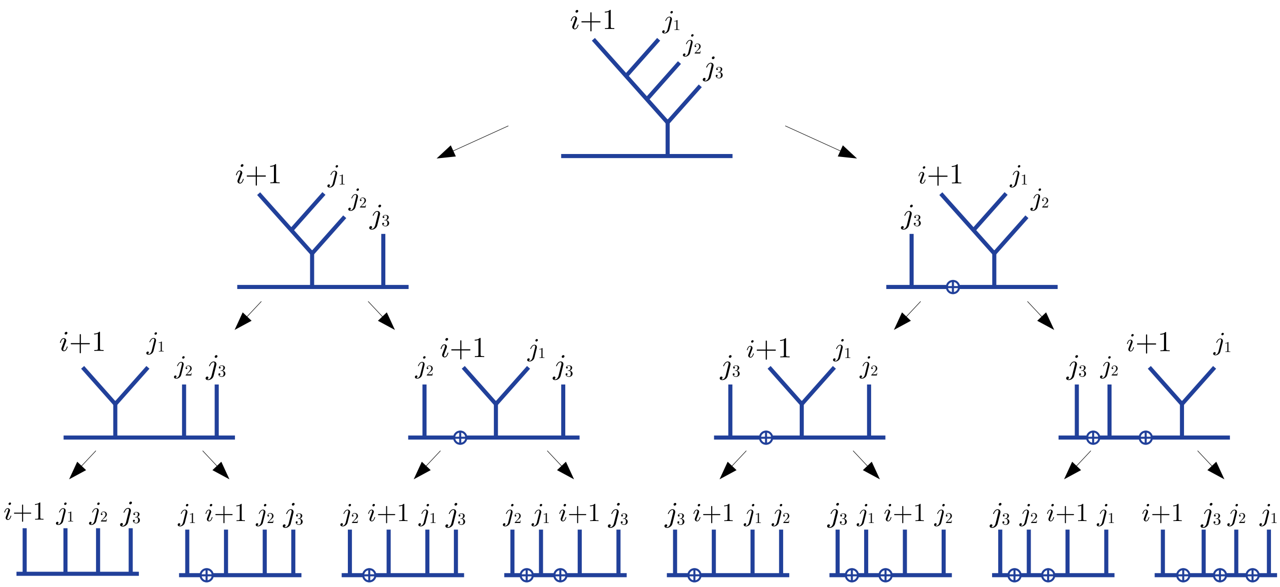


Figure 2.14: Iterated IHX relations.

on the Artin generators σ_1, σ_2:

$$\begin{array}{ll}
\gamma(\sigma_1)(1) = (2), & \gamma(\sigma_2)(1) = (1), \\
\gamma(\sigma_1)(2) = (1) + (12), & \gamma(\sigma_2)(2) = (3), \\
\gamma(\sigma_1)(3) = (3), & \gamma(\sigma_2)(3) = (2) + (23), \\
\gamma(\sigma_1)(12) = -(12), & \gamma(\sigma_2)(12) = (13), \\
\gamma(\sigma_1)(13) = (23), & \gamma(\sigma_2)(13) = (12) + (123) - (132), \\
\gamma(\sigma_1)(23) = (13) + (123), & \gamma(\sigma_2)(23) = -(23), \\
\gamma(\sigma_1)(123) = -(123), & \gamma(\sigma_2)(123) = (132), \\
\gamma(\sigma_1)(132) = -(123) + (132), & \gamma(\sigma_2)(132) = (123).
\end{array}$$

This gives us the following matrices:

$$\gamma(\sigma_1) = \left(\begin{array}{ccc|ccc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \quad \gamma(\sigma_2) = \left(\begin{array}{ccc|ccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0
\end{array}\right).$$

The global shape of these matrices was predicted by Theorem 2.3.5. Indeed in general we have the following.

Proposition 2.3.8. *For $\beta \in hB_n$ a homotopy braid, the matrix associated to $\gamma(\beta)$ in the basis of $\mathcal{V}$, endowed with the order resulting from Definition 1.2.8, is given by a lower triangular block matrix of the following form:*

$$\begin{pmatrix} B_{1,1} & 0 & \cdots & 0 \\ B_{2,1} & B_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & B_{n,2} & \cdots & B_{n,n} \end{pmatrix}$$

where $B_{i,i}$ is a finite order matrix of size $rk(\mathcal{V}_i) = \sum_{i-1}^{n-1} \frac{k!}{(k-i+1)!}$ which is the identity when β is pure. Moreover, $B_{1,1}$ corresponds to the left action by permutation $k \mapsto \pi^{-1}(\beta)(k)$, and $B_{2,2}$ corresponds to the left action on the set $\{(k,\, j)\}_{k<j}$ given by:

$$(k,\, j) \mapsto \begin{cases} \left(\pi^{-1}(\beta)(k),\, \pi^{-1}(\beta)(j)\right) & \text{if } \pi^{-1}(\beta)(k) < \pi^{-1}(\beta)(j), \\ -\left(\pi^{-1}(\beta)(j),\, \pi^{-1}(\beta)(k)\right) & \text{if } \pi^{-1}(\beta)(j) < \pi^{-1}(\beta)(k). \end{cases}$$

Proof. The triangular shape is a direct consequence of Theorem 2.3.5. Indeed, the chosen order respects the weight, and Theorem 2.3.5 shows that γ maps a commutator of weight k to a sum of commutators of weight at least k. Proposition 1.2.14 gives the size of the square diagonal blocks $B_{i,i}$. The fact that these diagonal blocks are the identity when β is a pure braid may require some more explanations. We only need to show this result on the generators $\beta = A_{i,j} = \mathbf{1}^{(i,j)}$. By Proposition 1.1.11, conjugating (α,∞) by (i,j) may only create a clasper (α',∞) of strictly higher degree. This shows that $\gamma(\beta)(\alpha) = (\alpha) +$ (strictly higher weight commutators) so that $B_{i,i}$ is the identity. The block matrix $B_{1,1}$ describes the action on degree one comb-claspers modulo claspers of higher degree: the claim follows on an easy verification on the generators σ_i. Similarly, the claim on the block matrix $B_{2,2}$ amounts to focusing on degree two comb-claspers. □

Remark 2.3.9. *Note that the blocks $B_{i,1}$ formed by the first n columns of the matrix encode the images of $\gamma(\beta)(x_i)$ on all the weight one commutators $x_1, \ldots, x_n$ of $\mathcal{V}$. In particular, these blocks encode the image of the homotopy Artin representation $\rho_h(\beta)(x_i)$ on all the generators $x_1, \ldots, x_n$ of $\mathcal{R}F_n$, and thus the full image of $\rho_h(\beta)$. Therefore, the n first columns of the matrix completely determine the full matrix $\gamma(\beta)$. Moreover, each block $B_{i,i}$ encodes the action of $\gamma(\beta)$ on weight i commutator up to higher weight commutators in $\mathcal{V}$. At the clasper level, this corresponds to the action on the degree i comb-claspers of the form (α,∞), up to claspers of higher degrees. According to Proposition 1.1.11, we can exchange clasper edges with other clasper edges or with strands of braids up to higher-degree claspers. This implies that each block $B_{i,i}$ is determined by the permutation $\pi(\beta)$ associated with the braid $\beta \in hB_n$.*

2.3.4 Injectivity

In order to prove the injectivity of γ, we need the following preparatory lemma.

Lemma 2.3.10. *Let $(i_1, \cdots, i_l)$ be a comb-clasper. We have*

$$\gamma\left(\mathbf{1}^{(i_1,\cdots,i_l)}\right)(i_l) = (i_l) - (i_1, \cdots, i_l),$$

where, on the right-hand side, $(i_1, \cdots, i_l)$ now denotes the corresponding commutator in $\mathcal{V}$.

Proof. Following the 3-steps procedure of Section 2.3.2, we consider the product

$$(i_1, \cdots, i_l)(i_l, \infty)(i_1, \cdots, i_l)^{-1}$$

and re-express it with only comb-claspers with ∞ in their support. To do so, as illustrated in Figure 2.15, we apply move (2) from Proposition 1.1.11 on the leaves on the i_l-th component, which introduces the comb-clasper $(i_1, \cdots, i_l, \infty)^{-1}$, and we simplify $(i_1, \cdots, i_l)$ and $(i_1, \cdots, i_l)^{-1}$. □

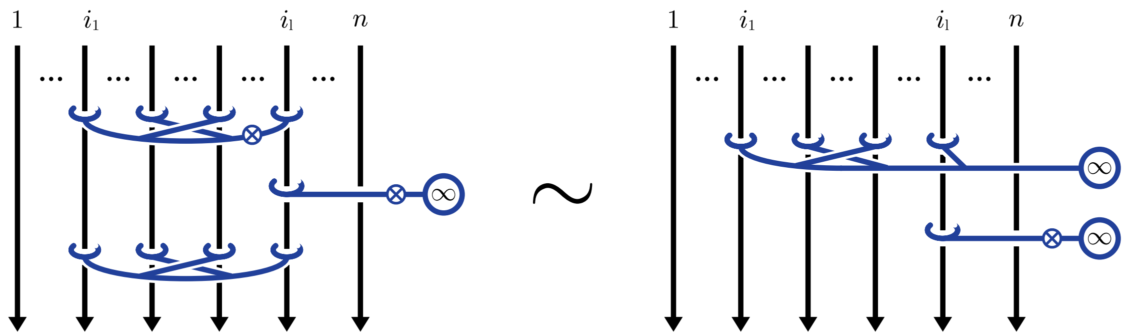


Figure 2.15: Proof of Lemma 2.3.10.

We can now state the injectivity of the representation γ from Proposition 2.3.2.

Theorem 2.3.11. *The representation $\gamma : hB_n \to GL(\mathcal{V})$ is injective.*

Proof. Let $\beta \in hB_n$ be such that $\gamma(\beta) = Id$. First, Proposition 2.3.8 imposes that β is a pure braid; indeed the block $B_{1,1}$ must be the identity, which means that the permutation $\pi(\beta)$ is trivial.

According to Theorem 2.1.11 we can consider a normal form for β:

$$\beta = \prod (\alpha)^{\nu_\alpha},$$

for some integers ν_α.

Let $I \subset \{1, \ldots, n\}$ be any sequence of indices. Let also $\mathcal{V}_I$ be the subspace of $\mathcal{V}$ spanned by commutators with support included in I. We can then define the associated projection $p_I : \mathcal{V} \to \mathcal{V}_I$, and its composition with the restriction of γ to $\mathcal{V}_I$, denoted by $\gamma_I := p_I \circ \gamma_{|_{\mathcal{V}_I}}$. Note that it corresponds to keeping only the components with index in I. It is clear using Proposition 1.1.11 that $\gamma(hP_n)(\mathcal{V}\backslash\mathcal{V}_I) \subset \mathcal{V}\backslash\mathcal{V}_I$, thus for $\beta_1, \beta_2 \in hP_n$ we have that $\gamma_I(\beta_1\beta_2) = \gamma_I(\beta_1)\gamma_I(\beta_2)$. Moreover $\gamma_I(\mathbf{1}^{(\alpha)}) = Id$ for any comb-clasper (α) with $\mathrm{supp}(\alpha) \not\subset I$. Hence $\gamma_I(\beta) = \gamma_I(\beta_I)$ for β_I defined by:

$$\beta_I = \prod_{\mathrm{supp}(\alpha) \subset I} (\alpha)^{\nu_\alpha}.$$

Now we show by strong induction on the degree of (α) that $\nu_\alpha = 0$. For the base case, we consider I of the form $I = \{i, l\}$ with $i \leqslant l$. Using Lemma 2.3.10 we obtain:

$$\begin{aligned} \gamma_I(\beta_I)(l) &= \gamma_I\left(\left(\mathbf{1}^{(il)}\right)^{\nu_{il}}\right)(l), \\ &= (l) - \nu_{il} \cdot (il). \end{aligned}$$

Since $\beta \in \ker(\gamma)$, we have that $\gamma_I(\beta)(l) = (l)$, and this implies that $\nu_\alpha = 0$ for any (α) of degree one. To prove that $\nu_\alpha = 0$ for any (α) of degree k we take I of length $k+1$ and using the strong induction hypothesis, we get then:

$$\beta_I = \prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu_\alpha}.$$

It is worth noting that for any comb-clasper (α) with support I and any commutator $(\alpha') \in \mathcal{V}_I$, we have $\gamma_I(\mathbf{1}^{(\alpha)})(\alpha') = (\alpha')$. This follows from the 3-steps procedure of Section 2.3.2 and Remark 1.1.12 given that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\alpha',\infty) \geqslant 2$. Thus thanks to Lemma 2.3.10, denoting by l the largest index of I we finally obtain:

$$\gamma_I(\beta_I)(l) = (l) - \sum_{\mathrm{supp}(\alpha)=I} \nu_\alpha \cdot (\alpha).$$

Because $\beta \in \ker(\gamma)$, we have that $\gamma_I(\beta)(l) = (l)$, and this implies $\nu_\alpha = 0$ for any (α) with support I. Repeating the argument for any $I \subset \{1, \ldots, n\}$ of length $k+1$, we get that $\nu_\alpha = 0$ for any (α) of degree k, which concludes the proof. □

Corollary 2.3.12. *The normal form is unique in hB_n, i.e., if $\beta = \prod(\alpha)^{\nu_\alpha} = \prod(\alpha)^{\nu'_\alpha}$ are two normal forms of β for a given order on the set of twisted comb-claspers, then $\nu_\alpha = \nu'_\alpha$ for any (α).*

Proof. The proof follows closely the previous one. As before for a given $I \subset \{1, \ldots, n\}$ we have $\gamma_I(\beta) = \gamma_I(\beta_I)$ for β_I defined by :

$$\beta_I = \prod_{\mathrm{supp}(\alpha)\subset I} (\alpha)^{\nu_\alpha} = \prod_{\mathrm{supp}(\alpha)\subset I} (\alpha)^{\nu'_\alpha}.$$

We show again by strong induction on the degree that $\nu_\alpha = \nu'_\alpha$ for all comb-claspers α. The base case is strictly similar, but for the inductive step one cannot in general write β_I with only comb-claspers with support I. However, by Remark 1.1.12, two comb-claspers (α) and (α') satisfying $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\alpha') \geqslant 2$ commute in hB_n. Hence, any comb-clasper with support equal to I commutes with any comb-clasper with support included in I. In particular, we get:

$$\begin{aligned}\gamma_I(\beta_I)(m) =& \gamma_I\left(\prod_{\mathrm{supp}(\alpha)\subsetneq I} (\alpha)^{\nu_\alpha}\right) \circ \gamma_I\left(\prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu_\alpha}\right)(m)\\ =& \gamma_I\left(\prod_{\mathrm{supp}(\alpha)\subsetneq I} (\alpha)^{\nu'_\alpha}\right) \circ \gamma_I\left(\prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu'_\alpha}\right)(m).\end{aligned}$$

Since comb-claspers (α) with $\mathrm{supp}(\alpha) \subsetneq I$ have degree $< k-1$ where k is the length of I, by induction hypothesis we have then,

$$\gamma_I\left(\prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu'_\alpha}\right)(m) = \gamma_I\left(\prod_{\mathrm{supp}(\alpha)=I} (\alpha)^{\nu'_\alpha}\right)(m).$$

By Lemma 2.3.10 we compute each term, thus obtaining:

$$(m) - \sum_{\mathrm{supp}(\alpha)=I} \nu_\alpha \cdot (\alpha) = (m) - \sum_{\mathrm{supp}(\alpha)=I} \nu'_\alpha \cdot (\alpha).$$

Clearly, the commutator family (α) with support equal to I is a free family in $\mathcal{V}$, so their coefficients ν_α and ν'_α on both sides coincide, which complete the induction and the proof. □

Remark 2.3.13. *Corollary 2.3.12 shows that the numbers ν_α of parallel copies of each comb-clasper in a normal form are a complete invariant of pure braids up to link-homotopy. We call those numbers the* **clasp-numbers**. *Other well known complete homotopy braid invariants are the* Milnor numbers *[Mil54]. As a matter of fact, Milnor numbers can be used, using the techniques of [Yas09], to give another proof of Corollary 2.3.12. In this thesis we will not try to make explicit the relation between clasp-numbers and Milnor numbers, since we work solely with clasp-numbers.*

2.4 A foretaste of the torsion problem

V. Lin in the Kourovka Notebook [MK14] asks the following: does the braid group B_n have proper non-abelian torsion-free factor-groups? P. Linnell and T. Schick in [LT07], give a positive answer to the question, showing that B_n is *residually torsion-free nilpotent-by-finite*. However their approach does not provide explicit examples. The homotopy braid groups hB_n appear as potential candidate. Indeed, S. P. Humphries shows in [Hum01] that hB_n is torsion-free for $n \leqslant 6$. In this section we extend this result to $n \leqslant 10$, using a new approach based on clasper calculus. Note that we will show the general result for all n later in the manuscript (see Section 4.3), using the broader context of *homotopy welded braids*. In this section, we focus on the study of torsion in the homotopy braid group, confining ourselves to their classical framework. Later, we will build on the results established in this section and extend them using the welded context.

2.4.1 Preparation

Throughout this section, we use the notion of normal form (Definition 2.1.8) in the pure homotopy braid group hP_n as a product of comb-claspers. To do this, we fix an order on the set of twisted comb-claspers, inspired by Definition 1.2.8. For two twisted comb-claspers $(\alpha) = (i_1 \cdots i_l)$ and $(\alpha') = (i'_1 \cdots i'_{l'})$ we set $(\alpha) \leqslant (\alpha')$ if:

- $\deg(\alpha) < \deg(\alpha')$, or
- $\deg(\alpha) = \deg(\alpha')$ and $i_1 \ldots i_l <_{\text{lex}} i'_1 \ldots i'_l$.

This order is used implicitly throughout the rest of the section.

Definition 2.4.1. *Let us take an integer k the equivalence relation generated by surgery along clasper of degree k and link-homotopy is called* $\mathbf{C_k}$***-homotopy***. *Given β and β' two braids we use the notation*

$$\theta \underset{C_k}{\overset{lh}{\sim}} \theta'$$

to mean that θ and θ' are C_k-homotopic.

Definition 2.4.2. *For two integers $k \leqslant n$ we define a projection map, $p_k : hP_n \to hP_n$ that sends a pure homotopy braid in normal form $\theta = \prod (\alpha)^{\nu_\alpha(\theta)}$ to its image $p_k(\theta) = \prod\limits_{\deg(\alpha)\leqslant k} (\alpha)^{\nu_\alpha(\theta)}$.*

Proposition 2.4.3. *Let θ,$\theta' \in hP_n$ be two pure homotopy braids, then for all $k \in \mathbf{N}$ the following assertions are equivalent:*

(i)
$$\theta \underset{C_{k+1}}{\overset{lh}{\sim}} \theta'$$

(ii)
$$\nu_\alpha(\theta) = \nu_\alpha(\theta'), \ \ \forall \deg(\alpha) \leqslant k,$$

(iii)
$$p_k(\theta) = p_k(\theta').$$

Proof. Let us first show that *(i)* implies *(ii)* i.e., for any comb-clasper (α) of degree k or less, we have $\nu_\alpha(\theta) = \nu_\alpha(\theta^T)$ for any clasper T for θ of degree $k+1$. To do so, we drag T by an isotopy along θ to re-express θ^T as the product $\theta \mathbf{1}^{T'}$ for some claspers T' for $\mathbf{1}$ of degree $k+1$. By lemma 2.1.7, there exist a product $(\alpha_1)\cdots(\alpha_m)$ of degree $k+1$ comb-claspers such that $\mathbf{1}^{T'} = (\alpha_1)\cdots(\alpha_m)$. Therefore,

$$\theta^T = \left(\prod(\alpha)^{\nu_\alpha(\theta)}\right)(\alpha_1)\cdots(\alpha_m),$$

with $\prod(\alpha)^{\nu_\alpha(\theta)}$ the normal form of θ. Starting with this expression, we apply the induction from the proof of Theorem 2.1.11, to get the normal form of θ^T. Note that in the process, we will only create claspers of degree greater than $k+1$, which does not change the value of the clasp-numbers ν_α with $\deg(\alpha) \leqslant k$.

Let us now prove that *(ii)* implies *(iii)*. We consider the normal forms $\theta = \prod(\alpha)^{\nu_\alpha(\theta)}$ and $\theta' = \prod(\alpha)^{\nu_\alpha(\theta')}$. It is clear that

$$p_k(\theta) = \prod_{\deg(\alpha)\leqslant k} (\alpha)^{\nu_\alpha(\theta)} = \prod_{\deg(\alpha)\leqslant k} (\alpha)^{\nu_\alpha(\theta')} = p_k(\theta')$$

if $\nu_\alpha(\theta) = \nu_\alpha(\theta')$ for all comb-claspers (α) with $\deg(\alpha) \leqslant k$.

Finally we conclude showing that *(iii)* implies *(i)*. Since C_{k+1}-moves allow to remove claspers of degree strictly higher than k, it is clear that θ is C_{k+1}-homotopic with its projection $p_k(\theta)$. So by transitivity if $p_k(\theta) = p_k(\theta')$ then θ and θ' are C_{k+1}-homotopic, and the proof is complete. □

Let us fix p a prime number. Let $\lambda \in hB_p$ be the homotopy braid illustrated in Figure 2.16, given by

$$\lambda = \sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}.$$

We denote by τ the cycle $(p\ p-1\ \cdots\ 2\ 1) = \pi(\lambda)$ associated to λ.

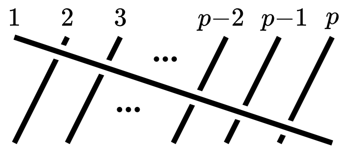


Figure 2.16: The homotopy braid λ.

Definition 2.4.4. *Let us denote by $\mathcal{O}$ a set of representative of the orbits for the action of τ^{-1} on the subsets of $\{1,\cdots,p\}$ $\big($i.e., $\tau^{-1}(\{i_1,\cdots,i_l\}) = \{\tau^{-1}(i_1),\cdots,\tau^{-1}(i_l)\}\big)$. We define $\mathcal{R}$ as the set of comb-claspers with support in $\mathcal{O}$ and $\mathcal{R}_k$ as the subset of degree k comb-claspers in $\mathcal{R}$. Finally, we order $\mathcal{R}$ with the order fixed above.*

Example 2.4.5. *Let illustrate Definition 2.4.4 with $p = 5$. The action of $\tau^{-1} := (12345)$ on the subset of $\{1,2,3,4,5\}$ contains 7 non-empty orbits. We choose a representative for each of them, thus obtaining:*

$$\mathcal{O} = \big\{\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\},\{1,2,3,4,5\}\big\}.$$

This gives us the following ordered set of comb-claspers:

$$\mathcal{R} = \big\{(12),(13),(123),(124),(1234),(1324),(12345),(12435),(13245),(13425),(14235),(14325)\big\}.$$

which is partitioned by the subsets $\mathcal{R}_1 = \big\{(12),(13)\big\}$, $\mathcal{R}_2 = \big\{(123),(124)\big\}$, $\mathcal{R}_3 = \big\{(1234),(1324)\big\}$ and $\mathcal{R}_4 = \big\{(12345),(12435),(13245),(13425),(14235),(14325)\big\}$.

Lemma 2.4.6. *For any comb-clasper (α), not necessarily in $\mathcal{R}$, and any pair $(\alpha_1),(\alpha_2) \in \mathcal{R}_k$ with $k \leqslant p-2$ we have the three following relations.*

(1) There exist some integers $l \in \mathbf{N}$ such that,

$$\lambda^l(\alpha)\lambda^{-l} = \prod_{(\alpha')\in\mathcal{R}_{\deg(\alpha)}} (\alpha').$$

(2) We have,

$$\nu_{\alpha_1}\left(\lambda^l(\alpha_2)\lambda^{-l}\right) = \begin{cases} 1 & \text{if } (\alpha_1) = (\alpha_2) \text{ and } l \equiv 0\ [p], \\ 0 & \text{if } (\alpha_1) \neq (\alpha_2) \text{ or } l \not\equiv 0\ [p]. \end{cases}$$

(3) For any integer $l \in \mathbf{N}$,

$$\nu_{1,\cdots,p}\left(\lambda^l(\alpha)\lambda^{-l}\right) = \begin{cases} 1 & \text{if } (\alpha) = (1,\cdots,p), \\ 0 & \text{if } (\alpha) \neq (1,\cdots,p). \end{cases}$$

Proof. Let us denote by i the isotopy sending $\lambda\mathbf{1}\lambda^{-1}$ to $\mathbf{1}$, then $\lambda(\alpha)\lambda^{-1} = i(\alpha)$ and $\mathrm{supp}(i(\alpha)) = \tau^{-1}(\mathrm{supp}(\alpha))$, thus for some integer l, the support $\mathrm{supp}\big(i^l(\alpha)\big)$ belongs to $\mathcal{O}$. Finally, using move *(8)* from Remark 1.1.13 and IHX relations, we turn $i^l(\alpha)$ into a product of comb-claspers and we get the first relation.

For the second one, if $l \not\equiv 0$ then $\mathrm{supp}\big(i^l(\alpha_2)\big) \notin \mathcal{O}$ and therefore $\nu_{\alpha_1}\big(i^l(\alpha_2)\big) = 0$. Moreover, it is clear that $\nu_{\alpha_1}(\alpha_2) = 0$ if $(\alpha_1) \neq (\alpha_2)$.

Finally, the first relation implies the third if $\deg(\alpha) \neq p-1$. Otherwise, let us consider the global shape of degree $p-1$ comb-claspers. If $(\alpha) = (1,\cdots,p-1,\cdots,i,p)$ with $i < p-1$ then by move *(8)* from Remark 1.1.13 and IHX relations, $i(\alpha)$ is given by a product of comb-claspers of the form $(1,i+1,\cdots,2,\cdots,p)$ as schematically illustrated in Figure 2.17.

If $(\alpha) = (1,\cdots,j,\cdots,i,\cdots,p-1,p)$ with $1 < i < j < p-1$ then by IHX relations, $i(\alpha)$ is given by a product of comb-claspers of the form $(1,\cdots,j+1,\cdots,i+1,\cdots,p)$ or $(1,\cdots,i+1,\cdots,2,\cdots,p)$. This fact is depicted in Figure 2.18. □

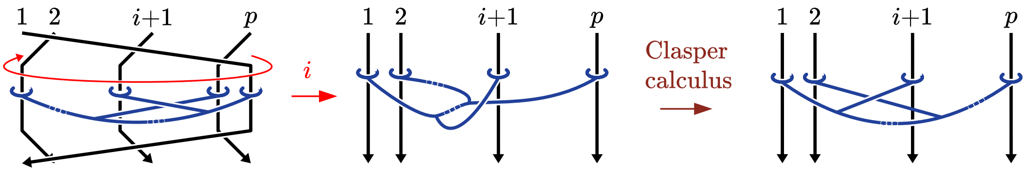


Figure 2.17: Computation of $\lambda(1,\cdots,p-1,\cdots,i,p)\lambda^{-1}$.

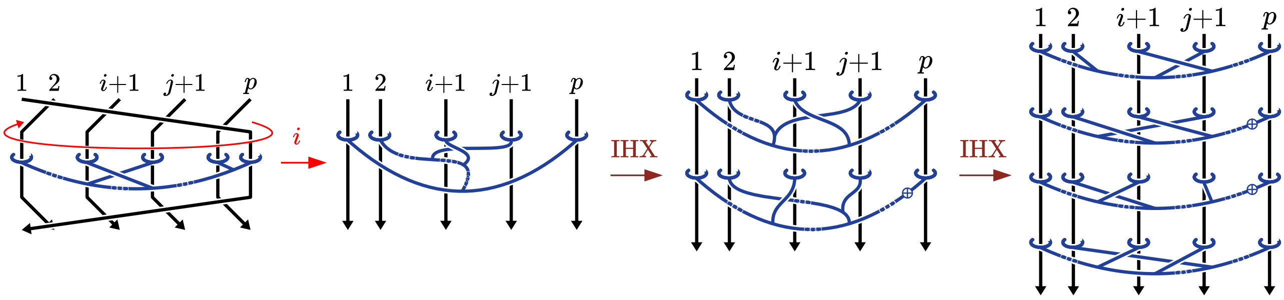


Figure 2.18: Computation of $\lambda(1,\cdots,j,\cdots,i,\cdots,p-1,p)\lambda^{-1}$.

Definition 2.4.7. *Let $\theta \in hP_p$ be a pure homotopy braid, we say that θ is in* **nice position** *if the normal form of θ satisfies:*

$$\prod(\alpha)^{\nu_\alpha(\theta)} = \prod_{(\alpha)\in\mathcal{R}}(\alpha)^{\nu_\alpha(\theta)}.$$

In other words we require that $\nu_\alpha(\theta) = 0$ if $(\alpha) \notin \mathcal{R}$.

Remark 2.4.8. *We emphasize that the normal form depends on the order on comb-claspers. Likewise, the property of being in nice position depends on the chosen order. Additionally, being in nice position also depends on the chosen set of orbit representatives $\mathcal{O}$.*

Lemma 2.4.9. *For any pure homotopy braid $\theta \in hP_p$ the product $\theta\lambda$ is conjugate to $\theta^*\lambda$, for some pure homotopy braid $\theta^* \in hP_p$ in nice position.*

Proof. Suppose that in the normal form of θ, the clasp-number $\nu_0 := \nu_{\alpha_0}(\theta)$ is not zero for some comb-clasper $(\alpha_0) \notin \mathcal{R}$. Let us further assume that (α_0) is of minimal degree, i.e., we have $\deg(\alpha_0) \leqslant \deg(\alpha)$ for all $(\alpha) \notin \mathcal{R}$ such that $\nu_\alpha(\theta) \neq 0$. Then according to equality *(1)* in Lemma 2.4.6, for some integer l, the conjugate $\lambda^l(\alpha_0)\lambda^{-l}$ is a product of comb-claspers in $\mathcal{R}$. We consider and compute the conjugate

$\theta'\lambda$ of $\theta\lambda$:

$$\begin{aligned}
\theta'\lambda &= \left(\prod_{0\leqslant k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right)^{-1} \theta\lambda \left(\prod_{0\leqslant k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right), \\
&= \left(\prod_{0\leqslant k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right)^{-1} \theta\lambda \left(\prod_{0\leqslant k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right) \lambda^{-1}\lambda, \\
&= \left(\prod_{0\leqslant k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right)^{-1} \theta \left(\prod_{0<k\leqslant l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right) \lambda, \\
&= \left(\prod_{0<k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right)^{-1} (\alpha_0)^{-\nu_0}\theta \left(\prod_{0<k<l} \lambda^k(\alpha_0)^{\nu_0}\lambda^{-k}\right) \left(\lambda^l(\alpha_0)^{\nu_0}\lambda^{-l}\right) \lambda.
\end{aligned}$$

Now note that, according to Lemma 2.4.6, the conjugates $\lambda^k(\alpha_0)\lambda^{-k}$ for $0 < k < l$ can be seen as products of comb-claspers with same degree as (α_0). Moreover thanks to moves *(2)* and *(4)* of Lemma 1.1.11, two comb-claspers commute up to claspers of higher degree, and by Lemma 2.1.7 we can assume that these higher degree claspers are also comb-claspers. Then in the previous expression, up to comb-claspers of degree greater than that of (α_0), we can simplify the terms $\lambda^k(\alpha_0)\lambda^{-k}$ for $0 < k < l$ with their inverse to obtain:

$$\theta' = (\alpha_0)^{-\nu_0}\theta \left(\lambda^l(\alpha_0)^{\nu_0}\lambda^{-l}\right) \left(\prod_{\deg(\alpha_0)<\deg(\alpha)} (\alpha)\right).$$

Since the factor $\alpha_0^{\nu_0}$ appears in the normal form $\theta = \prod(\alpha)^{\nu_\alpha(\theta)}$, we can, using the same argument, express θ' as follows:

$$\theta' = \left(\prod_{(\alpha)\neq(\alpha_0)} (\alpha)^{\nu_\alpha(\theta)}\right) \left(\lambda^l(\alpha_0)^{\nu_0}\lambda^{-l}\right) \left(\prod_{\deg(\alpha_0)<\deg(\alpha)} (\alpha)\right).$$

We recover the normal form of θ' using the same method as in proof of Theorem 2.1.11, rearranging claspers degree by degree. Let us compare the clasp-numbers of θ and θ'. First, if $\deg(\alpha) < \deg(\alpha_0)$ then $\nu_\alpha(\theta) = \nu_\alpha(\theta')$ since no claspers of degree lower than (α_0) appeared in the procedure. Second, it is clear that $\nu_{\alpha_0}(\theta') = 0$. Finally, $\nu_\alpha(\theta') = \nu_\alpha(\theta)$ for almost all others comb-claspers (α) of same degree as (α_0). The only exceptions come from the conjugate $\lambda^l(\alpha_0)^{\nu_0}\lambda^{-l}$ and involve comb-claspers belonging to $\mathcal{R}$.

In summary, for any comb-clasper $(\alpha) \notin \mathcal{R}$ of degree $\deg(\alpha) \leqslant \deg(\alpha_0)$ the clasp-numbers $\nu_\alpha(\theta')$ remain unchanged, except for $\nu_{\alpha_0}(\theta')$ which is now zero. Hence, by repeating the above argument, we eventually obtain another conjugate of $\theta\lambda$ of the form $\theta''\lambda$ satisfying $\nu_\alpha(\theta'') = 0$ for any comb-clasper $(\alpha) \notin \mathcal{R}$ such that $\deg(\alpha) \leqslant \deg(\alpha_0)$. Moreover, since above the degree p all claspers are trivial up to link-homotopy, using the same argument degree by degree, we will finally obtain a conjugate $\theta^*\lambda$ of $\theta\lambda$ with θ^* in nice position.

□

Lemma 2.4.10. *Let $\theta \in hP_p$ be in nice position. Then for any comb-clasper $(\alpha) \in \mathcal{R}_k$ with $k \leqslant p-2$ we have the following relations on the clasp-numbers:*

$$\nu_\alpha\big((\theta\lambda)^p\big) = \nu_\alpha\big((p_{k-1}(\theta)\lambda)^p\big) + \nu_\alpha(\theta).$$

Moreover for the comb-clasper $(\alpha) = (1, \cdots, p)$ we similarly have:

$$\nu_{1,\cdots,p}\big((\theta\lambda)^p\big) = \nu_{1,\cdots,p}\big((p_{p-2}(\theta)\lambda)^p\big) + p \times \nu_{1,\cdots,p}(\theta).$$

Proof. Since we ordered comb-claspers along their degree and since θ is in nice position, for any $k \leqslant p-1$, if we set

$$\delta_k := \prod_{(\alpha)\in\mathcal{R}_k} (\alpha)^{\nu_\alpha(\theta)},$$

then θ is C_{k+1}-homotopic with $p_{k-1}(\theta)\delta_k$. This gives us the following relation:

$$(\theta\lambda)^p = \left(\prod_{l=0}^{p-1} \lambda^l\theta\lambda^{-l}\right)\lambda^p \underset{C_{k+1}}{\overset{lh}{\sim}} \left(\prod_{l=0}^{p-1}\left(\lambda^l p_{k-1}(\theta)\lambda^{-l}\right)\left(\lambda^l\delta_k\lambda^{-l}\right)\right)\lambda^p.$$

To handle this expression and compute the clasp-number of $(\theta\lambda)^p$, we need the following claim.

Claim 2.4.11. *Let T be a degree $k \leqslant p-1$ clasper for the trivial braid and let $\Theta \in hP_p$ be a pure homotopy braid. Let also (α) be a degree k comb-clasper. Then,*

(1)

$$\Theta\mathbf{1}^T \underset{C_{k+1}}{\overset{lh}{\sim}} \mathbf{1}^T\Theta.$$

(2)

$$\nu_\alpha(\Theta\mathbf{1}^T) = \nu_\alpha(\Theta) + \nu_\alpha(\mathbf{1}^T).$$

Statement *(1)* is already true up to C_{k+1}-equivalence and follows from [Hab00b, Proposition 5.8]. By Lemma 2.1.7, the clasper T is given by a product of comb-claspers of degree k. Then, by statement *(1)*, up to C_{k+1}-homotopy, we can freely reposition these comb-claspers in the normal form of Θ. Therefore, using the implication *(i)* implies *(ii)* of Proposition 2.4.3 we deduce statement *(2)*.

For any integer l, the conjugate $\lambda^l\delta_k\lambda^{-l}$ is given by a union of claspers of degree k. Then, using statement *(1)* of the claim, we shift these claspers to the right in the above expression, and obtain:

$$(\theta\lambda)^p \underset{C_{k+1}}{\overset{lh}{\sim}} \left(\prod_{l=0}^{p-1}\left(\lambda^l p_{k-1}(\theta)\lambda^{-l}\right)\right)\lambda^p\left(\prod_{l=0}^{p-1}\lambda^l\delta_k\lambda^{-l}\right).$$

By simplifying the first product with λ^p, and writing δ_k as the product $\prod_{(\alpha)\in\mathcal{R}_k}(\alpha)^{\nu_\alpha(\theta)}$, we obtain:

$$(\theta\lambda)^p \underset{C_{k+1}}{\overset{lh}{\sim}} \big(p_{k-1}(\theta)\lambda\big)^p\left(\prod_{l=0}^{p-1}\prod_{(\alpha)\in\mathcal{R}_k}\left(\lambda^l(\alpha)\lambda^{-l}\right)^{\nu_\alpha(\theta)}\right).$$

For any comb-clasper $(\alpha) \in \mathcal{R}_k$ and any integer l, the conjugate $\lambda^l(\alpha)\lambda^{-l}$ is a clasper of degree k. Then, for any comb-clasper $(\alpha_0) \in \mathcal{R}_k$, using statement *(2)* repetitively we obtain the following equality:

$$\nu_{\alpha_0}\big((\theta\lambda)^p\big) = \nu_{\alpha_0}\Big(\big(p_{k-1}(\theta)\lambda\big)^p\Big) + \left(\sum_{l=0}^{p-1}\sum_{(\alpha)\in\mathcal{R}_k} \nu_\alpha(\theta)\nu_{\alpha_0}\left(\lambda^l(\alpha)\lambda^{-l}\right)\right).$$

Now, according to relation *(2)* of Lemma 2.4.6, if $k \leqslant p-2$, the only non-zero term in the sum is the factor $\nu_{\alpha_0}(\theta)\nu_{\alpha_0}(\alpha_0) = \nu_{\alpha_0}(\theta)$. This gives us the first equality of the lemma. Finally if $(\alpha_0) = (1,\cdots,p)$, by relation *(3)* of Lemma 2.4.6, for all l, we have $\nu_{\alpha_0}\left(\lambda^l(\alpha)\lambda^{-l}\right) = 1$ if $(\alpha) = (1,\cdots,p)$ and $\nu_{\alpha_0}\left(\lambda^l(1,\cdots,p)\lambda^{-l}\right) = 0$ otherwise. This gives us the second equality of the lemma. □

2.4.2 First results

Definition 2.4.12. *By induction, we construct a family $\{\theta_k\}_{k\leqslant p-2} \in hP_p$ of pure homotopy braids as follows:*

$$\begin{cases} \theta_0 & := \mathbf{1}, \\ \theta_{k+1} & := \theta_k \left(\displaystyle\prod_{(\alpha)\in\mathcal{R}_{k+1}} (\alpha)^{-\nu_\alpha\left((\theta_k\lambda)^p\right)}\right). \end{cases}$$

We emphasize that the construction of θ_{k+1} requires clasp-numbers of $(\theta_k\lambda)^p$, so it is necessary to compute its normal form.

Remark 2.4.13. *Since the order on $\mathcal{R}$ corresponds to the one on the set of all comb-claspers, and because we chose an order by increasing degree, we have that the family $\{\theta_k\}_{k\leqslant p-2} \in hP_p$ of pure homotopy braids are in nice position.*

Lemma 2.4.14. *Let $\theta \in hP_p$ be in nice position. If $(\theta\lambda)^p = \mathbf{1}$ then for any $k \leqslant p-2$, we have*

$$\theta \underset{C_{k+1}}{\overset{lh}{\sim}} \theta_k,$$

where $\{\theta_k\}_{k\leqslant p-2}$ is defined in Definition 2.4.12.

Proof. Firstly, thanks to (i) equivalent to (ii) in Proposition 2.4.3 and since θ and θ_k for any $k \leqslant p-2$ are in nice position, it is equivalent to show that for any $k \leqslant p-2$ and any $(\alpha) \in \mathcal{R}_k$,

$$\nu_\alpha(\theta) = \nu_\alpha(\theta_k).$$

We proceed by induction on the degree k of (α). Let us start with $(\alpha) \in \mathcal{R}_1$; by Lemma 2.4.10 we have

$$\nu_\alpha(\theta) = -\nu_\alpha(\lambda^p).$$

Moreover by construction we also have:

$$\nu_\alpha(\theta_1) = -\nu_\alpha(\lambda^p).$$

Thus $\nu_\alpha(\theta) = \nu_\alpha(\theta_1)$ for any $(\alpha) \in \mathcal{R}_1$ and the initialization step is done. For the induction we use Lemma 2.4.10 again and, for any $(\alpha) \in \mathcal{R}_k$ with $k \leqslant p-2$, we get the relation:

$$\nu_\alpha(\theta) = -\nu_\alpha\big((p_{k-1}(\theta)\lambda)^p\big).$$

Furthermore, by construction we also see that

$$\begin{aligned}\nu_\alpha(\theta_k) &= -\,\nu_\alpha\big((\theta_{k-1}\lambda)^p\big),\\ &= -\,\nu_\alpha\big((p_{k-1}(\theta_{k-1})\lambda)^p\big).\end{aligned}$$

Now by induction and (ii) equivalent to (iii) in Proposition 2.4.3, we have that $p_{k-1}(\theta) = p_{k-1}(\theta_{k-1})$ then $\nu_\alpha(\theta) = \nu_\alpha(\theta_k)$ for any $(\alpha) \in \mathcal{R}_k$, which concludes the proof. □

Lemma 2.4.15. *Let $\theta \in hP_p$ be in nice position. If $(\theta\lambda)^p = \mathbf{1}$ then*

$$\nu_{1,\cdots,p}\big((\theta_{p-2}\lambda)^p\big) \equiv 0 \mod [p],$$

where θ_{p-2} is defined in Definition 2.4.12.

Proof. Consider first the equality from Lemma 2.4.10:

$$\nu_{1,\cdots,p}((\theta\lambda)^p) = p \times \nu_{1,\cdots,p}(\theta) + \nu_{1,\cdots,p}((p_{p-2}(\theta)\lambda)^p).$$

By Lemma 2.4.14, θ and θ_{p-2} are C_{p-1}-homotopic. Hence by Proposition 2.4.3, we have

$$p_{p-2}(\theta) = p_{p-2}(\theta_{p-2}) = \theta_{p-2},$$

thus $\nu_{1,\cdots,p}((p_{p-2}(\theta)\lambda)^p) = \nu_{1,\cdots,p}((\theta_{p-2}\lambda)^p)$ and the above equality becomes:

$$\nu_{1,\cdots,p}((\theta_{p-2}\lambda)^p) = -p \times \nu_{1,\cdots,p}(\theta).$$

□

The following theorem is well-known, it appears for example in [HL90, Hum01]; we give here a new proof based on clasper calculus.

Theorem 2.4.16. *The pure homotopy braid group hP_n is torsion-free for any $n \in \mathbf{N}$.*

Proof. Let $\theta \in hP_n$ with $\theta \neq 1$ be a pure homotopy braid and let $k \in \mathbf{N}$ be the minimal integer such that θ is not C_{k+1}-homotopic to the trivial braid $\big($i.e., $k = \min\{l \in \mathbf{N} \mid p_l(\theta) \neq \mathbf{1}\}\big)$. Then we have:

$$\begin{aligned}\theta^m &\underset{C_{k+1}}{\overset{lh}{\sim}} \left(\prod_{deg(\alpha)=k} (\alpha)^{\nu_\alpha(\theta)}\right)^m\\ &\underset{C_{k+1}}{\overset{lh}{\sim}} \prod_{deg(\alpha)=k} (\alpha)^{m\nu_\alpha(\theta)},\end{aligned}$$

and θ^m is not C_{k+1}-homotopic to the trivial braid for any m. □

Lemma 2.4.17. *If there is torsion in hB_n then for some prime number $p \leqslant n$ there exists a torsion element of order p in hB_p.*

Proof. Let $\beta \in hB_n$ be a torsion element of prime order p and $\pi(\beta)$ its associated permutation. Now, by Theorem 2.4.16, hP_n is torsion-free, thus $\pi(\beta) \neq 1$ and $\pi(\beta)$ is a torsion element of order p in the symmetric group S_n. More precisely $\pi(\beta)$ is a product of p-cycles ($p \leqslant n$) with disjoint supports. Let us denote by $(i_1, \cdots, i_p)$ one of them, and by G the subgroup of hB_n generated by elements whose associated permutation sends the set $\{i_1, \cdots, i_p\}$ to itself. The homomorphism $\phi : G \to hB_p$, which keeps only the strands $i_1, \cdots, i_p$, sends β onto a torsion element of order p in hB_p and the proof is complete. □

Remark 2.4.18. *This lemma also holds for the usual braid group B_n and the proof works the same.*

Theorem 2.4.19. *There is no torsion in hB_n for $n \leqslant 10$.*

Proof. According to Lemma 2.4.17, we only need to show that for any prime number $p \leqslant 7$ there is no torsion elements of order p in hB_p. But, if such an element exists, it should be a conjugate of $\theta\lambda$, for some $\theta \in hP_p$, assumed to be in nice position by Lemma 2.4.9. We developed a program in *Python*, as presented in Appendix A and accessible on [Gra22], which constructs the family $\{\theta_k\}_{k \leqslant p-2}$ defined in Definition 2.4.12 for a given prime number p and returns $\nu_{1,\cdots,p}((\theta_{p-2}\lambda)^p)$. We ran it for $p = 2,3,5,7$ and each time $\nu_{1,\cdots,p}((\theta_{p-2}\lambda)^p) = 1$ so the condition of Lemma 2.4.15 does not hold and hB_n is torsion-free for $n \leqslant 10$. □

Remark 2.4.20. *It is likely that this method will enable us to extend the result to a larger number of strands. A more optimized program or greater computing power, would enable us to test the next prime numbers. However, it seems unlikely that a general result for any number of strands could be obtained this way. To demonstrate such a result, we need to consider the wider context of welded objects. This is done in Section 4.3, where we will reapply and adapt the ideas developed here to welded setting. Nevertheless, we have chosen to keep this first weak version in this manuscript, for the following two reasons. Firstly, this is the path we followed in our thesis work. We first addressed the torsion problem as presented in this section, then reconsidered it with welded techniques. Secondly, this illustrates the strength of welded theory. Indeed, we obtain a complete result while the same reasoning fails in the classical context.*

Chapter 3

Links up to link-homotopy

In this chapter, we will focus on the study of *links* up to link-homotopy. More precisely, we will describe in terms of *clasp-numbers variation* when two normal forms have link-homotopic *closures*.

The main purpose of this section is to use clasp-numbers, defined in Remark 2.3.13 above, to provide an explicit classification of links up to link-homotopy. In this way we recover results of J.W. Milnor [Mil54] and J. Levine [Lev88] for 4 or less components, and extend them partially for 5 components. To do so we first revisit in terms of claspers the work of N. Habegger and X.-S. Lin [HL90].

Remark 3.0.1. *Y. Kotorii and A. Mizusawa also addressed the question of using clasper theory to classify 4-component links up to link-homotopy in [KM20] and [KM22]. In their first paper, they use a different kind of normal form, arranged along a tetrahedron shape, adapted to the 4-component case. The main difference with the present work, however, is that their result makes direct use of Levine's classification. Here we instead reprove the latter using Theorem 3.1.8 and clasper calculus. Our approach is likely to extend to the general case: as an illustration of this fact, we treat the algebraically-split 5-component case in Section 3.2.4. Their other paper also also follows a similar direction. In this work, they ultimately adopted a similar technique and give a complete classification for links with at most 5 components.*

3.1 Habegger–Lin's work revisited

There is a procedure on braids called *closure*, that turns a braid into a link in S^3. The question is to determine when two braids have link-homotopic closures. The purpose of this section is to answer this question by following the work of [HL90] and reinterpreting it in terms of claspers. Let us first recall from [HL90, Theorem 1.7 & Corollary 1.11] that for any integer n we have the decomposition:

$$hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$$

where the first term corresponds to the braid obtained by omitting a given component, and the second term is the class of this component as an element of the reduced fundamental group of the disk with $n-1$ punctures.

In particular, if we iterate this decomposition by omitting the last component recursively, we obtain the decomposition illustrated in Figure 3.1 (see Convention 3.1.1) :

$$hP_n = \mathcal{R}F_1 \ltimes \cdots \ltimes \mathcal{R}F_{n-1}.$$

Moreover the normal form in hP_n with respect to the order of Definition 2.1.9 corresponds to this decomposition, where each individual factor is in normal form with respect to the order of Definition 1.2.8.

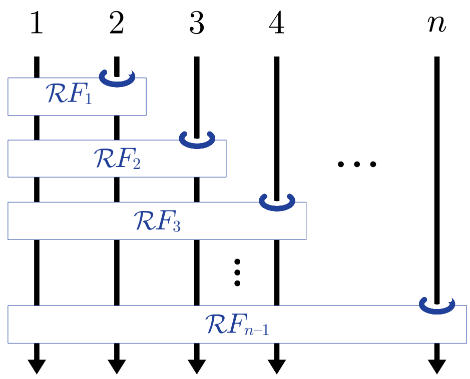
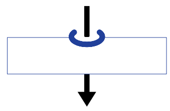


Figure 3.1: The Habegger–Lin decomposition in terms of clasper.

Convention 3.1.1. *In figures, a box intersecting several strands of* **1** *represents a product of claspers whose leaves may or may not intersect those strands, and are disjoint from all others strands. When each clasper in such a box intersects a given strand, this is shown by the graphical convention shown on the right (see Figures 3.1, 3.3, 3.4).*



To answer the question, N. Habegger and X.-S. Lin in [HL90] study an action of hP_{2n} on $hP_{n-1} \ltimes \mathcal{R}F_{n-1}$, which leads them to considering certain elementary operations $(\bar{x}_i,\bar{x}_i)_k$, $(x_i,x_i)_k$ and $(\bar{x}_i,x_i)_k$, whose definition we recall here in terms of claspers.

Definition 3.1.2. *Let $\beta \in hP_n$ be a pure homotopy braid, and let i, k be two distinct integers in $\{1, \dots, n\}$.*

- *$(\bar{x}_i,\bar{x}_i)_k(\beta)$ is the pure homotopy braid $\beta^{\Delta} \cdot \mathbf{1}^{(ik)^{-1}}$, where Δ and $(ik)^{-1}$ are degree one claspers as shown in the left-hand side of Figure 3.2.*
- *$(x_i,x_i)_k(\beta)$ is the pure homotopy braid $\mathbf{1}^{(ik)} \cdot \beta^{\Delta'}$, where Δ' and $(ik)^{-1}$ are degree one claspers as shown in the central part of Figure 3.2.*
- *$(\bar{x}_i,x_i)_k(\beta)$ is the pure homotopy braid $1^{(ik)}\beta \cdot \mathbf{1}^{(ik)^{-1}}$, where (ik) and $(ik)^{-1}$ are degree one claspers as shown in the right-hand side of Figure 3.2.*

Remark 3.1.3. *In fact, in [HL90] those operations are only defined for $k = n$, but the definitions extend naturally to any $k \neq i$. Moreover, Figure 2.8 in [HL90] does not correspond exactly to Figure 3.2, due to convention choices. Firstly, in [HL90] braids are oriented from bottom to top whereas we orient them from top to bottom. Secondly, here the basepoint of the second term in the decomposition $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$ is taken above the $n-1$ punctures, and not under the $n-1$ punctures as in [HL90].*

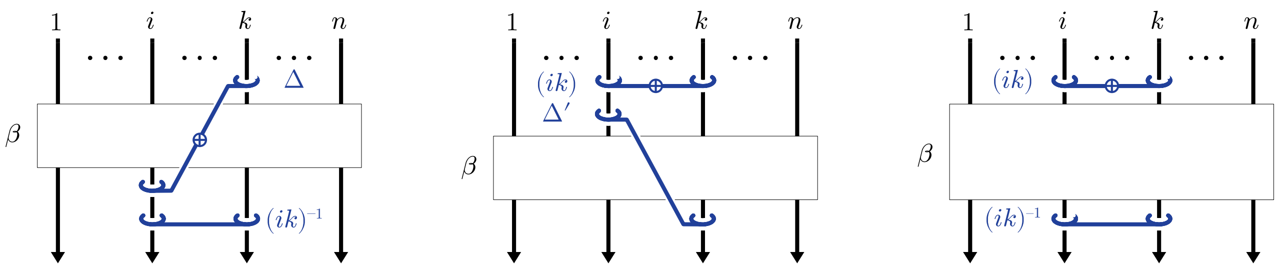


Figure 3.2: The elementary operations $(\bar{x}_i,\bar{x}_i)_k$, $(x_i,x_i)_k$, and $(\bar{x}_i,x_i)_k$.

By taking a closer look at the operations $(\bar{x}_i,\bar{x}_i)_k$ and $(\bar{x}_i,x_i)_k$ and more precisely their effect on the decomposition $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$, N. Habegger and X.-S. Lin come to the following central definition.

Definition 3.1.4. *Let $\beta \in hP_n$, we set $\beta = \theta\omega$ a decomposition in $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$. A* **partial conjugate** *of β is an element of hP_n of the form $\theta\lambda\omega\lambda^{-1}$ for some $\lambda \in \mathcal{R}F_{n-1}$. We speak of a k-th* partial conjugation*, or* partial conjugation with respect to the k-th component*, when the decomposition $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$ is obtained by omitting the k-th component.*

The computations in [HL90, p. 413] show that the operations $(\bar{x}_i,\bar{x}_i)_k$ and $(\bar{x}_i,x_i)_k$ are partial conjugations. We use clasper calculus to reprove it for the operation $(\bar{x}_i,\bar{x}_i)_k$ in Proposition 3.1.5 and later for the operation $(\bar{x}_i,x_i)_k$ in Proposition 3.1.7.

Proposition 3.1.5. *Let β be a pure homotopy braid. The operation $(\bar{x}_i,\bar{x}_i)_k(\beta)$ is the k-th partial conjugation of β by x_i. In particular the operations $(\bar{x}_i,\bar{x}_i)_k$ with $i \neq k$ in $\{1, \ldots, n\}$ generate the partial conjugations.*

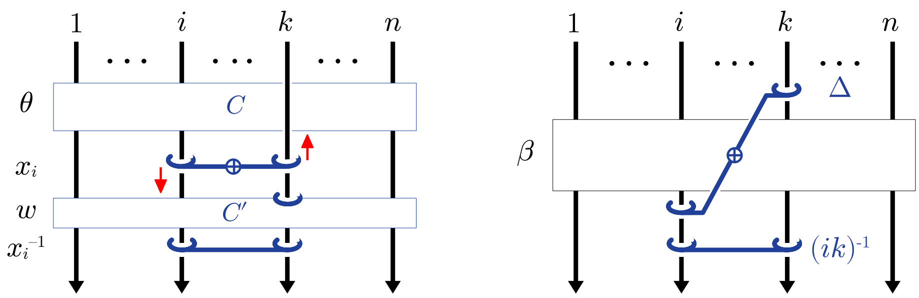


Figure 3.3: The k-th partial conjugation by x_i.

Proof. We set first $\beta = \theta\omega$ the decomposition of β in $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$ obtained by omitting the k-th component. Through surgery, we see the factor $\theta \in hP_{n-1}$ as a union C of simple claspers for the trivial braid $\mathbf{1}$, where the k-th component is disjoint from and passes over all claspers in C. The factor $\omega \in \mathcal{R}F_{n-1}$ is given by a union C' of simple claspers for the trivial braid, all containing k in their support. In this setting, the k-th partial conjugation by x_i (i.e., $\beta \mapsto \theta x_i \omega x_i^{-1}$) corresponds to the product $C(ik)C'(ik)^{-1}$ as shown in the left-hand side of Figure 3.3. To prove the proposition it suffices to slide the leaf k of (ik) upwards by an isotopy (this is possible since C is disjoint from the k-th component), and slide the leaf i downwards: by moves (2) and (4) from Proposition 1.1.11 this creates claspers with repeats, which by Lemma 1.1.10 are trivial up to link-homotopy. ☐

J. R. Hughes in [Hug05] showed that partial conjugations generate conjugations. We reprove this result below using clasper calculus.

Proposition 3.1.6. *Partial conjugations generate conjugations, in other words operations* $(\bar{x}_i,\bar{x}_i)_k$ *generate operations* $(\bar{x}_i,x_i)_k$ *for* $i \neq k$ *in* $\{1, \ldots, n\}$.

Proof. It suffices to show that partial conjugations generate all conjugations by any comb-clasper (ik). Let $\beta \in hP_n$, seen as the surgery on $\mathbf{1}$ along a union of simple claspers denoted C. By the procedure given below, we decompose C into a product $C \sim \tilde{C}C_iC_kC_{i,k}$ such that:

- $C_{i,k}$ is a union of claspers each having i and k in their support,
- C_i, resp C_k, is a union of claspers, each having i, resp k, in their support, and such that the k-th, resp i-th, component of $\mathbf{1}$ is disjoint from and passes over all claspers in C_i, resp C_k,
- $\tilde{C}$ is a product of claspers that are disjoint from and pass under the i-th and k-th components.

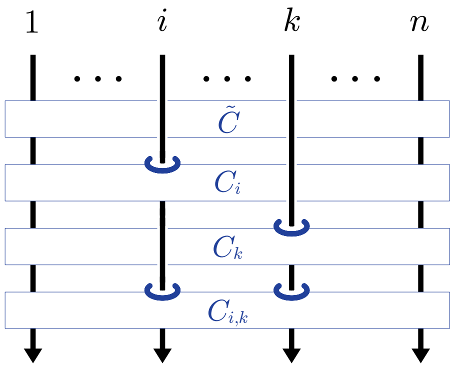


Figure 3.4: Decomposition $C \sim \tilde{C}C_iC_kC_{i,k}$.

This decomposition is illustrated in Figure 3.4. To obtain such a decomposition, we first consider those claspers in C that are disjoint from the i-th and k-th components, and we apply move *(3)* from Proposition 1.1.11 to ensure that they all are behind those components. We use moves *(2)* and *(4)* from Proposition 1.1.11 to obtain a decomposition $C \sim \tilde{C}C_0$ where all claspers in C_0 have either i or k in their support. Next, we consider those claspers in C_0 that are disjoint from the k-th component: we apply move *(3)* from Proposition 1.1.11 to ensure that they all are behind this component, and then use again Proposition 1.1.11 to obtain a decomposition $C \sim \tilde{C}C_iC_1$ where all claspers in C_1 have k in their support. Finally, by the exact same way we have a decomposition $C_1 \sim C_kC_{i,k}$ with C_k and $C_{i,k}$ as desired.

Note that the product $(\tilde{C}C_i)(C_kC_{i,k})$ corresponds to the decomposition $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$ given by omitting the k-th component. We can then apply a k-th partial conjugation by x_i to obtain $\tilde{C}C_i(ik)C_kC_{i,k}(ik)^{-1}$. We then exchange the relative position of (ik) with C_k using moves *(2)* and

(4) from Proposition 1.1.11, this creates a union $K_{i,k}$ of claspers with i and k in their support, such that:

$$(ik)C_k = C_k K_{i,k}(ik). \tag{3.1}$$

We can then freely (up to link-homotopy) exchange (ik) and $C_{i,k}$ by Remark 1.1.12, thus obtaining the decomposition $\tilde{C}C_iC_kK_{i,k}C_{i,k}$. Now we similarly use moves *(2)* and *(4)* from Proposition 1.1.11 to exchange C_i and C_k, which creates a union $R_{i,k}$ of claspers with i and k in their support, such that:

$$C_iC_k = C_kR_{i,k}C_i. \tag{3.2}$$

We obtain in this way the product $(\tilde{C}C_k)(R_{i,k}C_iK_{i,k}C_{i,k})$ corresponding to the decomposition $hP_n = hP_{n-1} \ltimes \mathcal{R}F_{n-1}$ given by omitting the i-th component. We can then perform an i-th partial conjugation by x_k to obtain $\tilde{C}C_k(ik)R_{i,k}C_iC_{i,k}K_{i,k}(ik)^{-1}$ that is link-homotopic to $\tilde{C}C_k(ik)R_{i,k}C_iC_{i,k}(ik)^{-1}K_{i,k}$ according to Remark 1.1.12. Then with further partial conjugations, we relocate $K_{i,k}$ and we obtain $\tilde{C}C_k(ik)K_{i,k}R_{i,k}C_iC_{i,k}(ik)^{-1}$. Finally using equality (3.1) and (3.2) from above we simplify the expression as follows:

$$\tilde{C}C_kK_{i,k}(ik)R_{i,k}C_iC_{i,k}(ik)^{-1} \sim \tilde{C}(ik)C_kR_{i,k}C_iC_{i,k}(ik)^{-1} \sim \tilde{C}(ik)C_iC_kC_{i,k}(ik)^{-1},$$

and we conclude by exchanging $\tilde{C}$ and (ik) via an isotopy, thus obtaining the conjugate $(ik)C(ik)^{-1}$. □

Proposition 3.1.7. *The operations $(\bar{x_i},\bar{x_i})_k$ generate the operations $(\bar{x_i},x_i)_k$ for $i \neq k$ in $\{1, \ldots, n\}$.*

Proof. Clearly, from the clasper point of view, the operation $(x_i,x_i)_k$ is the composition of the inverse of the operation $(\bar{x_k},\bar{x_k})_i$ with the conjugation by the comb-clasper (ik) (i.e., the operation $(\bar{x_i},x_i)_k$). Hence we conclude the proof using Proposition 3.1.6. □

We state now the main classification theorem of links up to link-homotopy.

Theorem 3.1.8. *[HL90, Hug05] Let β, $\beta' \in hP_n$ be two pure homotopy braids. The closures of β and β' are link-homotopic, if and only if there exists a sequence $\beta = \beta_0, \beta_1, \ldots, \beta_n = \beta'$ of elements of hP_n such that $\beta_{j+1} = (\bar{x_i},\bar{x_i})_k(\beta_j)$ for some $i \neq k$ in $\{1, \ldots, n\}$.*

Proof. Firstly, [HL90, Theorem 2.13.] states that β and β' have link-homotopic closures if and only if there exists a sequence $\beta = \beta_0, \beta_1, \cdots, \beta_n = \beta'$ of elements of hP_n such that β_{i+1} is a conjugate, or a partial conjugate of β_i. Moreover, as mentioned above (Proposition 3.1.5) the operations $(\bar{x_i},\bar{x_i})_k$ generate the partial conjugations, and we conclude the proof using the result from [Hug05] (see Proposition 3.1.6). □

3.2 Link-homotopy classification

This section is dedicated to the explicit classification of links up to link-homotopy. The starting point of the strategy is Theorem 3.1.8 which allows us to see links up to link-homotopy as pure homotopy braids up to operations $(\bar{x_i},\bar{x_i})_k$ with $i \neq k$ in $\{1, \ldots, n\}$. Moreover, with Corollary 2.3.12 we show that a braid is uniquely determined by its normal form, encoded by a sequence of integers: the clasp-numbers. The goal is then to determine how the normal form, or equivalently the clasp-numbers, vary under operations $(\bar{x_i},\bar{x_i})_k$. By using clasper calculus, we recover in this way the link-homotopy classification results from J. W. Milnor [Mil54] and J. Levine [Lev88] in the case of links with at most 4 components. We then apply these techniques to the 5-component *algebraically-split* case.

In order to use Corollary 2.3.12, we need to fix an order on the set of twisted comb-claspers. In the rest of the document, we fix the following order, which is inspired from Definition 1.2.8. For two twisted comb-claspers $(\alpha) = (i_1 \cdots i_l)$ and $(\alpha') = (i'_1 \cdots i'_{l'})$ we set $(\alpha) \leqslant (\alpha')$ if:

- $\deg(\alpha) < \deg(\alpha')$, or
- $\deg(\alpha) = \deg(\alpha')$ and $i_1 \dots i_l <_{\text{lex}} i'_1 \dots i'_l$.

This order is used implicitly throughout the rest of the document.

3.2.1 The 2-component case.

As a warm-up, we consider the 2-component case in order to illustrate the techniques of this section.

Let L be a 2-component link, then L can be seen as the closure of a 2-component string-link β. As mentioned in Remark 2.1.5, up to link-homotopy, string-links correspond to pure braid. Thus by Corollary 2.3.12 there is a unique integer ν_{12} such that:

$$\beta \sim (12)^{\nu_{12}}.$$

So by Theorem 3.1.8 the link-homotopy class of L is uniquely characterized by the integer ν_{12} modulo the indeterminacy introduced by the operations $(\bar{x_2},\bar{x_2})_1$ and $(\bar{x_1},\bar{x_1})_2$.

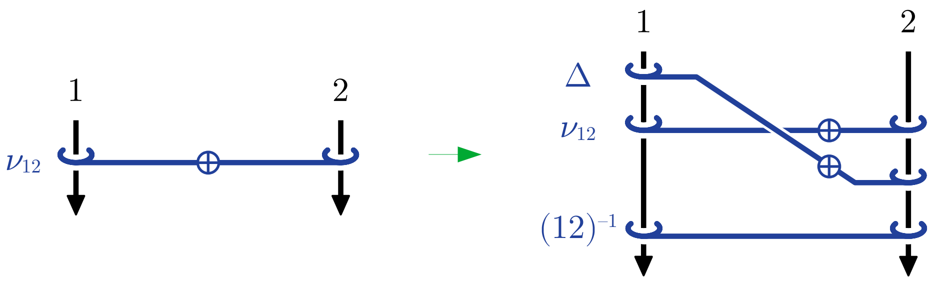


Figure 3.5: Operation $(\bar{x_2},\bar{x_2})_1$ on the 2-component normal form.

However, in both cases, $|\text{supp}(\Delta) \cap \text{supp}(12)| = 2$ as illustrated in the right-hand side of Figure 3.5 for $(\bar{x_2},\bar{x_2})_1$. Thus, Remark 1.1.12 shows that $(\bar{x_2},\bar{x_2})_1$ and $(\bar{x_1},\bar{x_1})_2$ leave the normal form unchanged, and the clasp-number ν_{12} is therefore a complete link-homotopy invariant for 2-component links. Note that this number is in fact the linking number between the two components, which is well known to classify 2-component links up to link-homotopy.

3.2.2 The 3-component case

Let L be a 3-component link seen as the closure of the normal form:

$$(12)^{\nu_{12}}(13)^{\nu_{13}}(23)^{\nu_{23}}(123)^{\nu_{123}},$$

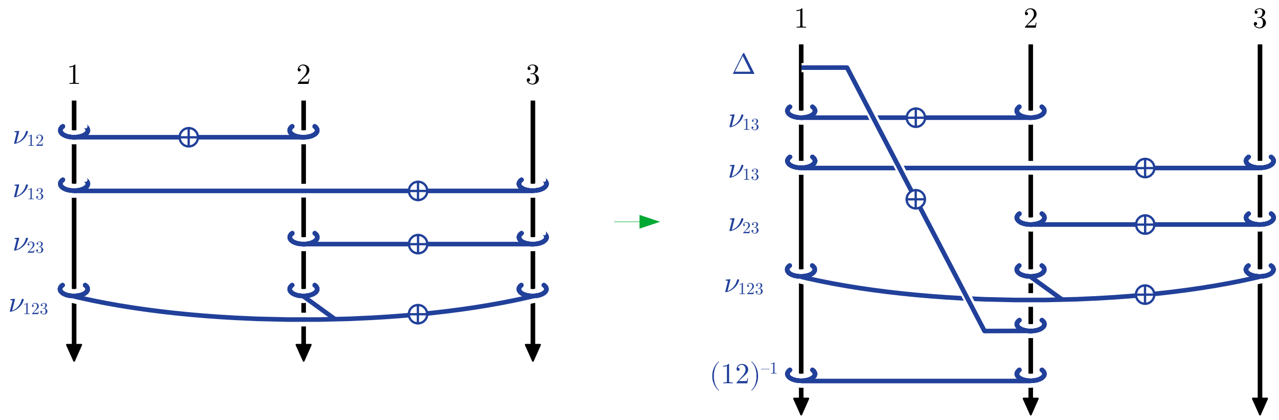


Figure 3.6: Operation $(\bar{x_2},\bar{x_2})_1$ on the 3-component normal form.

for some integers ν_{12}, ν_{13}, ν_{23} and ν_{123}. See the left-hand side of Figure 3.6.

We now investigate how these numbers vary under the operations $(\bar{x_i},\bar{x_i})_k$ for $i \neq k \in \{1, 2, 3\}$; we apply for example $(\bar{x_2},\bar{x_2})_1$. By Definition 3.1.2 this corresponds to introducing the claspers Δ and $(12)^{-1}$ as shown in the right-hand side of Figure 3.6, which we then put in normal form. This is done by sliding the 1-leaf of Δ along the first component to obtain (12) and simplify it with $(12)^{-1}$. By move (2) from Proposition 1.1.11, this sliding creates new claspers, but by Lemma 1.1.10, the only claspers that do not vanish up to link-homotopy, are those created when Δ crosses the leaves of $(13)^{\nu_{13}}$: more precisely, in this process, ν_{13} copies of $\{1, 2, 3\}$-supported claspers appear. Finally, according to Remark 1.1.12 we can rearrange these new claspers and the normal form becomes

$$(12)^{\nu_{12}}(13)^{\nu_{13}}(23)^{\nu_{23}}(123)^{\nu_{123}+\nu_{13}}.$$

The other operations $(\bar{x_i},\bar{x_i})_k$ act in a similar way, by changing ν_{123} by a multiple of ν_{12}, ν_{13} or ν_{23}. Summarizing, we have shown that

$$\nu_{12},\ \nu_{13},\ \nu_{23} \text{ and } \nu_{123} \bmod \gcd(\nu_{12}, \nu_{13}, \nu_{23}),$$

form a set of complete invariants for 3-component links up to link-homotopy.

Note that we recover here Milnor invariants $\overline{\mu}_{12}$, $\overline{\mu}_{13}$, $\overline{\mu}_{23}$ and $\overline{\mu}_{123}$, that we already knew to be complete link-homotopy invariants for 3-component links (see [Mil54]).

3.2.3 The 4-component case

Before proceeding with the link-homotopy classification of 4-component links, we need the following technical result.

Lemma 3.2.1. *Let C be a union of simple claspers for the trivial n-component braid $\mathbf{1}$, and let $l \in \{1, \ldots, n\}$. Let T be a clasper in C with l in its support and let $C_{T,l} = \bigcup T'$ be the union of all claspers T' in C such that* $\mathrm{supp}(T') \cap \mathrm{supp}(T) = \{l\}$. *Suppose that an l-leaf f of T is disjoint from a 3-ball B containing all l-leaves of $C_{T,l}$. Then the closure of $\mathbf{1}^C$ is link-homotopic to the closure of $\mathbf{1}^{C'}$ where C' is obtained from C by passing f across the ball B as shown in Figure 3.7.*

Proof. First the result is clear if T has several l-leaves, since by Lemma 1.1.10, T vanishes up to link-homotopy. By Remark 1.1.12 the edges of any clasper in $C_{T,l}$ can freely cross those of T but f and the l-leaves of claspers in $C_{T,l}$ cannot be freely exchanged. However, according to Remark 1.1.12 again, the leaf f can be freely exchanged with any l-leaf of claspers in $C \backslash C_{T,l}$, since their support contain at least some $k \neq l$ which is in $\mathrm{supp}(T)$. By using the closure we can thus slide f in the other direction, using the closure of $\mathbf{1}$, and bypass the l-leaves of claspers in $C_{T,l}$ all gathered in B. □

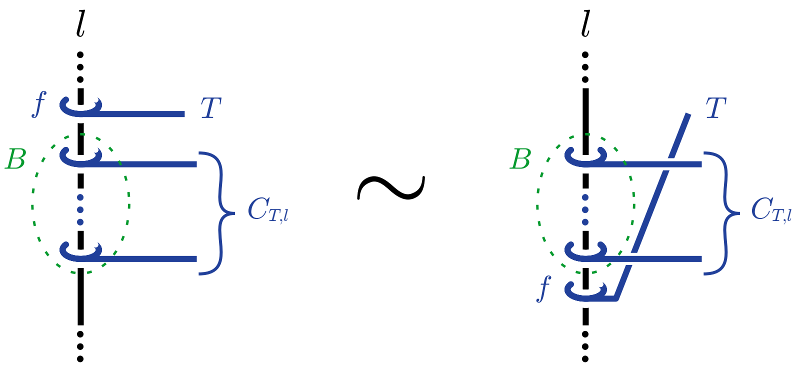


Figure 3.7: Illustration of Lemma 3.2.1

Although the assumption of Lemma 3.2.1 may seem restrictive, it turns out to be naturally satisfied for normal forms. For instance, we have the following consequence.

Proposition 3.2.2. *Let* $C = (\alpha_1)^{\nu_1} \cdots (\alpha_m)^{\nu_m}$ *be the normal form of a pure homotopy* n*-component braid and let* (α) *be a degree* $n-2$ *comb-clasper. Then* C *and* $C' = (\alpha_1)^{\nu_1} \cdots (\alpha)(\alpha_i)^{\nu_i}(\alpha)^{-1} \cdots (\alpha_m)^{\nu_m}$ *have link-homotopic closures, for any* $i \in \{1, \ldots, m\}$.

Proof. We first consider the product $(\alpha_1)^{\nu_1} \cdots (\alpha_i)^{\nu_i}(\alpha)(\alpha)^{-1} \cdots (\alpha_m)^{\nu_m}$ where we just insert the trivial term $(\alpha)(\alpha)^{-1}$ in C. We next want to exchange (α) and $(\alpha_i)^{\nu_i}$. This is allowed if $|\mathrm{supp}(\alpha) \cap \mathrm{supp}(\alpha_i)| \geqslant 2$ by Remark 1.1.12, but if $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\alpha_i) = \{l\}$ we can only realize crossing changes between the edges of (α) and $(\alpha_i)^{\nu_i}$ (see Remark 1.1.12). However in that case (α_i) is a comb-clasper of support $\{k, l\}$ with k the only component not in the support of (α), thus we can apply Lemma 3.2.1 to the l-leaf of (α), and bypass the block $(\alpha_i)^{\nu_i}$ (corresponding to C_T in Lemma 3.2.1). □

Let us now return to the classification of links up to link-homotopy and let L be a 4-component link seen as the closure of the normal form:

$$(12)^{\nu_{12}}(13)^{\nu_{13}}(14)^{\nu_{14}}(23)^{\nu_{23}}(24)^{\nu_{24}}(34)^{\nu_{34}}(123)^{\nu_{123}}(124)^{\nu_{124}}(134)^{\nu_{134}}(234)^{\nu_{234}}(1234)^{\nu_{1234}}(1324)^{\nu_{1324}},$$

for some integers ν_{12}, ν_{13}, ν_{14}, ν_{23}, ν_{24}, ν_{34}, ν_{123}, ν_{124}, ν_{134}, ν_{234}, ν_{1234}, and ν_{1324}. See Figure 3.8.

We can apply Proposition 3.2.2 to the degree 2 comb-claspers (123), (124), (134) and (234). For example, applying Proposition 3.2.2 to $(\alpha) = (234)$ and $(\alpha_i) = (12)$, we get that L is link-homotopic to the closure of:

$$(234)(12)^{\nu_{12}}(234)^{-1}(13)^{\nu_{13}}(14)^{\nu_{14}}(23)^{\nu_{23}}(24)^{\nu_{24}}(34)^{\nu_{34}}(123)^{\nu_{123}}$$
$$(124)^{\nu_{124}}(134)^{\nu_{134}}(234)^{\nu_{234}}(1234)^{\nu_{1234}}(1324)^{\nu_{1324}}.$$

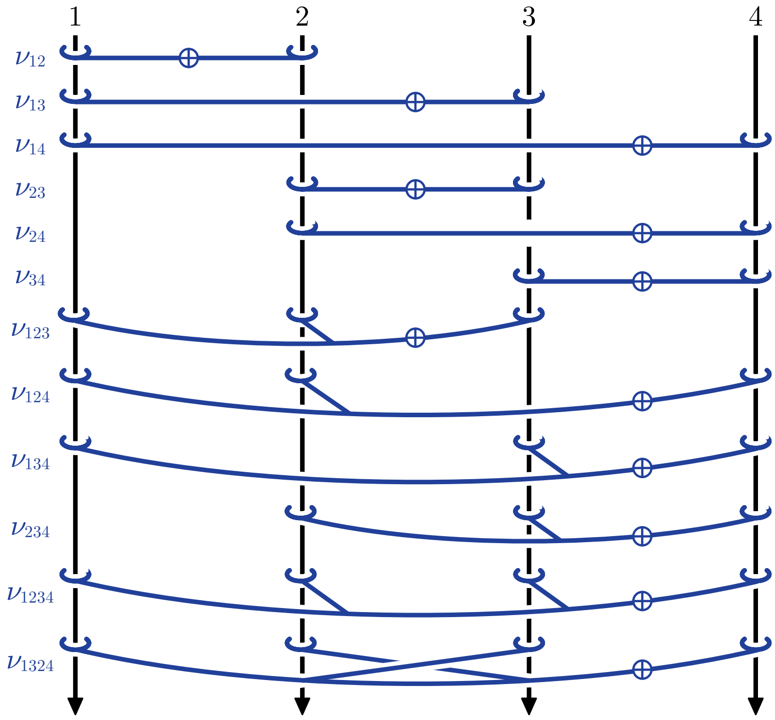


Figure 3.8: Normal form for 4 components.

By clasper calculus (Proposition 1.1.11 and Remark 1.1.12), we have that $(234)(12)^{\nu_{12}}(234)^{-1}$ is link-homotopic to $(12)^{\nu_{12}}(1234)^{\nu_{12}}$. The product of claspers $(1234)^{\nu_{12}}$ can be freely homotoped by Remark 1.1.12, thus producing the normal form

$$(12)^{\nu_{12}}(13)^{\nu_{13}}(14)^{\nu_{14}}(23)^{\nu_{23}}(24)^{\nu_{24}}(34)^{\nu_{34}}(123)^{\nu_{123}}(124)^{\nu_{124}}(134)^{\nu_{134}}(234)^{\nu_{234}}(1234)^{\nu_{1234}+\nu_{12}}(1324)^{\nu_{1324}},$$

whose closure is link-homotopic to L. This is recorded in the first row of Table 3.1, which records all possible transformations on clasp-numbers obtained with Proposition 3.2.2. Each row represents a possible transformation, where the entry in the column ν_α represents the variation of the clasp-number ν_α. Note that an empty cell means that the corresponding clasp-number remains unchanged. Note also that, we only need two columns because for the comb-claspers of degree 1 or 2 the associated clasp-numbers remain unchanged.

ν_{1234}	ν_{1324}
ν_{12}	
ν_{34}	
	ν_{13}
	ν_{24}
ν_{14}	$-\nu_{14}$
ν_{23}	$-\nu_{23}$

Table 3.1: Some clasp-numbers variation with same closures.

Let us now describe how operations $(\bar{x_i},\bar{x_i})_k$ for $i \neq k$ in $\{1, \ldots, 4\}$ affect clasp-numbers. As for the 3-component case, $(\bar{x_i},\bar{x_i})_k$ corresponds to sliding the i-leaf of a simple clasper of support $\{i, j\}$ (denoted Δ in Definition 3.1.2) along the i-th component. Along the way Δ encounters leaves and edges of other claspers, that can be crossed as described by moves (2) and (4) from Proposition 1.1.11. In doing so, claspers of degree 2 and 3 may appear, that we must reposition in the normal form. Those of degree 3 commute with any clasper by Remark 1.1.12, but since they may not be comb-claspers we have to use IHX relations (Proposition 1.1.15) to turn them into comb-claspers. Claspers of degree 2 can be repositioned using Remark 1.1.12 and Lemma 3.2.1 (the fact that Lemma 3.2.1 applies is clear according to the shape of the normal form, where factors are stacked).

We detail as an example operation $(\bar{x_4},\bar{x_4})_2$. In that case Δ has support $\{2, 4\}$ and we slide its 2-leaf along the 2nd component. According to Remark 1.1.12, Δ can freely cross the edges of claspers with 4 in their support and the 2-leaves of claspers containing 2 and 4 in their support. Thus, we only consider the claspers that appear when Δ meets the edges of $(13)^{\nu_{13}}$ and the 2-leaves of $(12)^{\nu_{12}}$, $(23)^{\nu_{23}}$ and $(123)^{\nu_{123}}$. Once repositioned we obtain in order the factors $(1324)^{\nu_{13}}$, $(124)^{\nu_{12}}$, $(234)^{-\nu_{23}}$ and $(1324)^{-\nu_{123}}$. However according to Table 3.1, $(1324)^{\nu_{13}}$ can be removed up to link-homotopy and thus we get the following normal form:

$$(12)^{\nu_{12}}(13)^{\nu_{13}}(14)^{\nu_{14}}(23)^{\nu_{23}}(24)^{\nu_{24}}(34)^{\nu_{34}}(123)^{\nu_{123}}(124)^{\nu_{124}+\nu_{12}}$$
$$(134)^{\nu_{134}}(234)^{\nu_{234}-\nu_{23}}(1234)^{\nu_{1234}}(1324)^{\nu_{1324}-\nu_{123}}.$$

In the same way, we compute all operations $(\bar{x_i},\bar{x_i})_k$ and record them in Table 3.2. The entry in row $(\bar{x_i},\bar{x_i})_k$ represents the corresponding variation. As in Table 3.1, an empty cell means that $(\bar{x_i},\bar{x_i})_k$ does not change the clasp-number. Moreover the ν_{ik} columns are omitted because they remain unchanged by any operations.

	ν_{123}	ν_{124}	ν_{134}	ν_{234}	ν_{1234}	ν_{1324}
$(\bar{x_2},\bar{x_2})_1$	ν_{13}	ν_{14}			ν_{134}	
$(\bar{x_3},\bar{x_3})_1$	$-\nu_{12}$		ν_{14}			ν_{124}
$(\bar{x_4},\bar{x_4})_1$		$-\nu_{12}$	$-\nu_{13}$		$-\nu_{123}$	ν_{123}
$(\bar{x_1},\bar{x_1})_2$	$-\nu_{23}$	$-\nu_{24}$			$-\nu_{234}$	
$(\bar{x_3},\bar{x_3})_2$	ν_{12}			ν_{24}	ν_{124}	$-\nu_{124}$
$(\bar{x_4},\bar{x_4})_2$		ν_{12}		$-\nu_{23}$		$-\nu_{123}$
$(\bar{x_1},\bar{x_1})_3$	ν_{23}		$-\nu_{34}$			ν_{234}
$(\bar{x_2},\bar{x_2})_3$	$-\nu_{13}$			$-\nu_{34}$	$-\nu_{134}$	ν_{134}
$(\bar{x_4},\bar{x_4})_3$			ν_{13}	ν_{23}	ν_{123}	
$(\bar{x_1},\bar{x_1})_4$		ν_{24}	ν_{34}		ν_{234}	$-\nu_{234}$
$(\bar{x_2},\bar{x_2})_4$		$-\nu_{14}$		ν_{34}		$-\nu_{134}$
$(\bar{x_3},\bar{x_3})_4$			$-\nu_{14}$	$-\nu_{24}$	$-\nu_{124}$	

Table 3.2: Clasp-numbers variations under operations $(\bar{x_i},\bar{x_i})_k$.

There are however algebraic redundancies in Table 3.2, i.e., some lines are combinations of other lines, which means that some operation $(\bar{x_i},\bar{x_i})_k$ generate the others. So we can keep only these ones (or their opposite), which we call *'generating'* operations, and which we record in Table 3.3.

ν_{123}	ν_{124}	ν_{134}	ν_{234}	ν_{1234}	ν_{1324}
ν_{13}	ν_{14}			ν_{134}	
$-\nu_{12}$		ν_{14}			ν_{124}
ν_{23}	ν_{24}			ν_{234}	
	$-\nu_{12}$		ν_{23}		ν_{123}
ν_{23}		$-\nu_{34}$			ν_{234}
		ν_{13}	ν_{23}	ν_{123}	
	ν_{14}		$-\nu_{34}$		ν_{134}
		ν_{14}	ν_{24}	ν_{124}	

Table 3.3: Clasp-numbers variations under generating operations.

Finally, with Table 3.3 we reinterpret the homotopy classification of 4-component links as follows.

Theorem 3.2.3. *Two 4-component links, seen as closures of braids in normal forms (see Figure 3.8), are link-homotopic if and only if their clasp-numbers are related by a sequence of transformations from Table 3.3.*

Remark 3.2.4. *Table 3.1 was only used here as a tool to simplify the computations summarized in Table 3.2. We stress that Table 3.3 alone suffices to generate Table 3.1 and Table 3.2. In particular, Table 3.1 is obtained by 'commuting' the rows of Table 3.3. More precisely let us denote by $[R_i]_k$ the variation associated to the i-th row of Table k. Let us also denote by $[R_i,R_j]_k$ the 'commutator of rows i and j' from Table k, i.e., the variation obtained by applying the i-th row of Table k, then the j-th, then the opposite of the i-th and finally the opposite of the j-th. Thus, Table 3.3 generates the rows of Table 3.1 as follows:*

$$[R_1]_{3.1} = [R_6,R_2]_{3.3}, \qquad [R_2]_{3.1} = [R_1,R_5]_{3.3}, \qquad [R_3]_{3.1} = [R_6,R_7]_{3.3},$$
$$[R_4]_{3.1} = [R_3,R_2]_{3.3}, \qquad [R_5]_{3.1} = [R_2,R_1]_{3.3}, \qquad [R_6]_{3.1} = [R_5,R_6]_{3.3}.$$

Note that J. Levine in [Lev88] already proved a similar result. The purpose of this paragraph is to explain the correspondence between the two approaches. The strategy adopted in [Lev88] consists in fixing the first three components and let the fourth one carry the information of the link-homotopy indeterminacy. J. Levine used four integers k, l, r, d to describe a normal form for the first three components, and integers e_i with $i \in \{1, \ldots, 8\}$ to describe the information of the last component. Finally, in [Lev88, Table3] he gives a list of all possible transformations on e_i-numbers that do not change the link-homotopy class. Fixing the last component corresponds in our setting to fixing the clasp-number ν_{123}: this is why [Lev88, Table 3] has one less column than Tables 3.2 and 3.3. Moreover, the five rows of [Lev88, Table 3] correspond to $(\bar{x_3},\bar{x_3})_1^{-1}$,$(\bar{x_4},\bar{x_4})_2^{-1}$,$(\bar{x_1},\bar{x_1})_4$, $(\bar{x_3},\bar{x_3})_4$ and $(\bar{x_1},\bar{x_1})_2^{-c} \circ (\bar{x_3},\bar{x_3})_1^{-a} \circ (\bar{x_2},\bar{x_2})_1^{-b}$, respectively, and Levine's integers correspond to clasp-numbers as follows.

k	r	l	d	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
ν_{12}	ν_{13}	ν_{23}	ν_{123}	ν_{14}	ν_{24}	ν_{34}	ν_{124}	ν_{134}	ν_{234}	$-\nu_{1324}$	$-\nu_{1234}$

3.2.4 The 5-component algebraically-split case

This section is dedicated to the study of *5-components algebraically-split links*. These are links such that the linking number is zero for any pair of components. Equivalently, algebraically-split links are given by the closure of a normal form with trivial clasp-numbers for any degree one comb-clasper.

The following proposition is the algebraically-split version of Proposition 3.2.2. The proof is essentially the same and is left to the reader.

Proposition 3.2.5. *Let $C = (\alpha_1)^{\nu_1} \cdots (\alpha_m)^{\nu_m}$ be a normal form of a pure homotopy n-component braid with $\nu_i = 0$ for any (α_i) of degree one, and let (α) be a degree $n-3$ comb-clasper. Then C and $C' = (\alpha_1)^{\nu_1} \cdots (\alpha)(\alpha_i)^{\nu_i}(\alpha)^{-1} \cdots (\alpha_m)^{\nu_m}$ have link-homotopic closures, for any $i \in \{1, \ldots, m\}$.*

Now, let L be a 5-component algebraically-split link seen as the closure of the normal form:

$$\begin{aligned} C =&(123)^{\nu_{123}}(124)^{\nu_{124}}(125)^{\nu_{125}}(134)^{\nu_{134}}(135)^{\nu_{135}}(145)^{\nu_{145}}(234)^{\nu_{234}}(235)^{\nu_{235}}(245)^{\nu_{245}}(345)^{\nu_{345}}\\ &(1234)^{\nu_{1234}}(1235)^{\nu_{1235}}(1245)^{\nu_{1245}}(1324)^{\nu_{1324}}(1325)^{\nu_{1325}}(1345)^{\nu_{1345}}(1425)^{\nu_{1425}}(1435)^{\nu_{1435}}(2345)^{\nu_{2345}}\\ &(2435)^{\nu_{2435}}(12345)^{\nu_{12345}}(12435)^{\nu_{12435}}(13245)^{\nu_{13245}}(13425)^{\nu_{13425}}(14235)^{\nu_{14235}}(14325)^{\nu_{14325}}. \end{aligned}$$

The strategy is similar to the 4-component case. We see links as braid closures, and with Theorem 2.3.12 we know that any braid is uniquely determined up to link-homotopy by a set of clasp-numbers $\{\nu_\alpha\}$. In this case, the algebraically-split condition results in the vanishing of clasp-numbers ν_{ij} (i.e., $\nu_\alpha = 0$ for all (α) of degree 1). Now, as mentioned by Theorem 3.1.8, the classification of links up to link-homotopy reduces to determining how operations $(\bar{x_i},\bar{x_i})_k$ for $i \neq k$ in $\{1, \ldots, 5\}$ affect the clasp-numbers.

We first use Proposition 3.2.5 to simplify the upcoming computations. In that case Proposition 3.2.5 concerns degree 2 comb-claspers (123), (124), (125), (134), (135), (145), (234), (235), (245) and (345). We record in Table 3.4 all possible transformations on clasp-numbers obtained with Proposition 3.2.5. As before, each row represents a possible transformation, where the entry in the column ν_α represents the variation of the clasp-number ν_α, and an empty cell means that the corresponding clasp-number remains unchanged. Note also that we only need columns corresponding to degree 4 comb-claspers because the other clasp-numbers remain unchanged.

Finally, we compute the effect of all operations $(\bar{x_i},\bar{x_i})_k$ using Definition 3.1.2 and Table 3.4, and simplify the results keeping only the *'generating'* operations, as in the 4-component case. We record the corresponding clasp-number variations in Table 3.5. As for the 4-component case, Table 3.5 contains all the data for the classification of 5-component algebraically-split links. In other words we obtain the following classification result.

Theorem 3.2.6. *Two 5-component algebraically-split links, seen as closures of braids in normal forms, are link-homotopic if and only if their clasp-numbers are related by a sequence of transformations from Table 3.5.*

Remark 3.2.7. *Just as in Remark 3.2.4, only Table 3.5 is needed here as it generates Table 3.4. With the same notations as in Remark 3.2.4 and with the additional notation '$\circ$' for composition, we get:*

ν_{12345}	ν_{12435}	ν_{13245}	ν_{13425}	ν_{14235}	ν_{14325}
ν_{123}					
		ν_{123}			
	ν_{124}				
				ν_{124}	
ν_{125}	$-\nu_{125}$				
			ν_{125}		$-\nu_{125}$
			ν_{134}		
					ν_{134}
	ν_{135}			$-\nu_{135}$	
		ν_{135}	$-\nu_{135}$		
ν_{145}		$-\nu_{145}$			
				ν_{145}	$-\nu_{145}$
ν_{234}	$-\nu_{234}$		$-\nu_{234}$		ν_{234}
	ν_{234}	$-\nu_{234}$	ν_{234}	$-\nu_{234}$	
				ν_{235}	
					ν_{235}
		ν_{245}			
			ν_{245}		
ν_{345}					
	ν_{345}				

Table 3.4: Some clasp-numbers variations with same closure.

ν_{1234}	ν_{1235}	ν_{1245}	ν_{1324}	ν_{1325}	ν_{1345}	ν_{1425}	ν_{1435}	ν_{2345}	ν_{2435}	ν_{12345}	ν_{12435}	ν_{13245}	ν_{13425}	ν_{14235}	ν_{14325}
ν_{134}	ν_{135}	ν_{145}								ν_{1345}	ν_{1435}				
			ν_{124}	ν_{125}	ν_{145}							ν_{1245}	ν_{1425}		
$-\nu_{123}$			ν_{123}			ν_{125}	ν_{135}							ν_{1235}	ν_{1325}
ν_{234}	ν_{235}	ν_{245}								ν_{2345}	ν_{2435}				
		ν_{125}	$-\nu_{123}$			$-\nu_{125}$			ν_{235}		ν_{1235}	ν_{1325}	$-\nu_{1325}$	$-\nu_{1235}$	
				ν_{123}		ν_{124}		ν_{234}	$-\nu_{234}$				$\nu_{1234}+\nu_{1324}$		$-\nu_{1234}$
			ν_{234}	ν_{235}	$-\nu_{345}$							$\nu_{2345}+\nu_{2435}$	$-\nu_{2435}$		
ν_{134}	ν_{135}		$-\nu_{134}$	$-\nu_{135}$				ν_{345}		ν_{1345}		$-\nu_{1345}$		ν_{1435}	$-\nu_{1435}$
	$-\nu_{123}$						ν_{134}		ν_{234}		$\nu_{1234}+\nu_{1324}$			$-\nu_{1324}$	
ν_{234}			$-\nu_{234}$			ν_{245}	ν_{345}							$\nu_{2345}+\nu_{2435}$	$-\nu_{2345}$
ν_{124}					ν_{145}		$-\nu_{145}$	ν_{245}	$-\nu_{245}$	ν_{1245}	$-\nu_{1245}$		ν_{1425}		$-\nu_{1425}$
		ν_{124}			ν_{134}			ν_{234}		ν_{1234}		ν_{1324}			
				ν_{135}		ν_{145}		$-\nu_{345}$	ν_{345}				ν_{1345}		ν_{1435}
	ν_{125}						ν_{145}		ν_{245}		ν_{1245}			ν_{1425}	
		ν_{125}			ν_{135}			ν_{235}		ν_{1235}		ν_{1325}			

Table 3.5: Clasp-numbers variations under generating operations in the 5-component algebraically-split case.

$$[R_1]_{3.4} = [R_{12},R_3]_{3.5} \circ [R_5,R_6]_{3.5}, \quad [R_2]_{3.4} = [R_6,R_5]_{3.5}, \quad [R_3]_{3.4} = [R_{11},R_{12}]_{3.5},$$
$$[R_4]_{3.4} = [R_6,R_{14}]_{3.5}, \quad [R_5]_{3.4} = [R_5,R_{11}]_{3.5} \circ [R_3,R_{11}]_{3.5}, \quad [R_6]_{3.4} = [R_3,R_{11}]_{3.5},$$
$$[R_7]_{3.4} = [R_{12},R_{13}]_{3.5}, \quad [R_8]_{3.4} = [R_8,R_9]_{3.5}, \quad [R_9]_{3.4} = [R_1,R_5]_{3.5},$$
$$[R_{10}]_{3.4} = [R_{13},R_5]_{3.5}, \quad [R_{11}]_{3.4} = [R_2,R_1]_{3.5}, \quad [R_{12}]_{3.4} = [R_{13},R_{14}]_{3.5},$$
$$[R_{13}]_{3.4} = [R_6,R_4]_{3.5}, \quad [R_{14}]_{3.4} = [R_7,R_9]_{3.5}, \quad [R_{15}]_{3.4} = [R_5,R_{10}]_{3.5},$$
$$[R_{16}]_{3.4} = [R_7,R_3]_{3.5}, \quad [R_{17}]_{3.4} = [R_4,R_2]_{3.5}, \quad [R_{18}]_{3.4} = [R_{10},R_{11}]_{3.5},$$
$$[R_{19}]_{3.4} = [R_1,R_7]_{3.5}, \quad [R_{20}]_{3.4} = [R_{10},R_1]_{3.5}.$$

Note that in a recent paper [KM22], Kotorii and Mizusawa, with techniques similar to the one presented in this section, have given a complete classification of 5-component links up to link-homotopy.

Chapter 4

Welded objects

This chapter deals with welded objects. The structure is very similar to that of Chapters 1 and 2. First, general definitions are given in Section 4.1, including a review of *arrow calculus*, which is the welded analogue of clasper calculus, developed by J.-B. Meilhan and A. Yasuhara in [MY19]. Section 4.2 is devoted to the *homotopy welded braids group*. We give in Theorem 4.2.11 a group presentation inspired by that of J. Darné [Dar23]. We then extend the representation of Section 2.3 to the welded framework. Finally, Section 4.3 takes up the elements developed in Section 2.4 from the welded point of view. We end with the main result of this thesis, namely Theorem 4.3.8: the homotopy braid group is torsion-free for any number of strands.

4.1 General definitions

4.1.1 Virtual diagrams

This section focuses on the study of *welded tangles*, a generalization of the usual tangles previously studied.

Definition 4.1.1. *An* n-component **virtual tangle diagram** *is the image of a smooth immersion of an n-component, ordered, and oriented* 1*–manifold (a disjoint union of circles and intervals) in the disk. We require the embedding to be proper, meaning that the boundary of the* 1*-manifold must be sent to the boundary of the disk. Additionally, we require the singularities to consist of a finite number of transverse double points labeled either as a classical crossing or as a virtual crossing, as illustrated in Figure 4.1.*

Figure 4.1: A classical and a virtual crossing.

In what follows, we will often simply say 'diagram' instead of virtual tangle diagram.

Definition 4.1.2. *An* n-component **welded tangle** *is the equivalence class of an n-component virtual tangle diagram under* **welded isotopies** *given by:*

- *planar isotopies,*
- *classical Reidemeister moves,*
- *virtual Reidemeister moves, which are the exact analogues of the classical ones with all classical crossings replaced by virtual ones,*
- *mixed Reidemeister move, as shown on the left-hand side of Figure 4.2,*
- *OC moves (for overcrossings commute), as shown in the central part of Figure 4.2.*

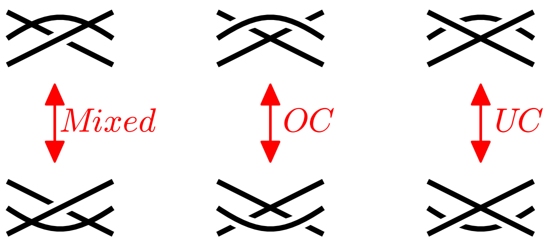


Figure 4.2: The Mixed, OC and UC moves on virtual diagrams.

Remark 4.1.3. *Recall that there is a 'forbidden' local moves, called* UC moves *(for undercrossings commute), illustrated on the right-hand side of Figure 4.2. Recall also that the notion of* ***virtual tangle*** *arises by deleting the OC move from Definition 4.1.2 [Kau99, GPV00].*

Remark 4.1.4. *It is shown in [Kau99] that the set of tangles up to isotopy is injected into the set of welded tangle up to welded isotopy. In other words, if two classical tangles are related by welded isotopy, then they are also related by classical isotopy.*

As in the context of classical knot theory, we can study the notion of **link-homotopy**, where each individual component is allowed to cross itself. In the welded context, however, it turns out that the right analogue of this relation is generated by the **self-virtualization** move, where a crossing involving two strands of the same component can be replaced by a virtual one or vice versa, as depicted in Figure 4.3 see [ABMW17a]. In what follows, we will study this equivalence relation and call it *homotopy*; we use the same notation as in the classical case '$\sim$' to denote this equivalence relation.

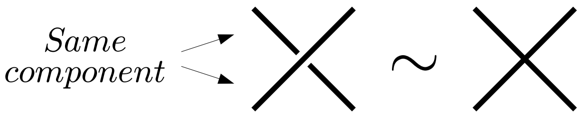


Figure 4.3: A self-virtualization move.

4.1.2 Arrow calculus

Arrow calculus has been developed by J.-B. Meilhan and A. Yasuhara in [MY19]. It is the analogue of claspers calculus in the welded context. In particular, it turns out to be a powerful tool to deal with homotopy. In the following we define and recall the *homotopy arrow calculus*.

Definition 4.1.5. *[MY19, Definition 3.1] A* **w–tree** *for a diagram D is a connected uni-trivalent tree T, immersed in the plane of the diagram such that*

- *The trivalent vertices of T are pairwise disjoint and disjoint from D.*
- *The univalent vertices of T are pairwise disjoint and are contained in $D \backslash \{\text{crossings of } D\}$.*
- *All edges of T are oriented, such that each trivalent vertex has two ingoing edges and one outgoing edge.*
- *We allow virtual crossings between edges of T, and between D and edges of T, but classical crossings involving T are not allowed.*
- *Each edge of T is assigned a number (possibly zero) of decorations, called* **twists**, *which are disjoint from all vertices and crossings.*

*A w–tree with a single edge is called a w–*arrow.

The unique univalent vertex with an ingoing edge is called the **head** of the w-tree. By graphic convention, it is represented by an arrow on the figures. The other univalent vertices are called **tails**. When we do not need to distinguish between *tails* and *head*, we simply call all univalent vertices, ***endpoints***. In the figures, portions of the diagram are represented by thick black lines and w-trees edges by thin blue lines. Finally, twists are represented graphically by big red dots •.

Definition 4.1.6. *Let T be a w-tree for a diagram D. We define the degree of T, denoted by* $\deg(T)$*, as its number of tails. The* **support** *of T, denoted by* $\operatorname{supp}(T)$*, is defined to be the set of the components of D that intersect the endpoints of T. The roots of T, denoted by* $\operatorname{roots}(T)$*, is defined to be the set of the components of D that intersect the tails of T. We will often consider the number of the components rather than the components themselves.*

Definition 4.1.7. *We say that a w-tree for a diagram D has* **repeats** *if it intersects a component of D in at least two endpoints. Otherwise, we say that the w-tree is* **nonrepeated**.

Given a disjoint union of w-trees F for a diagram D, there is a procedure called surgery detailed in [MY19] to construct a new diagram denoted D_F. We illustrate on Figure 4.4 the surgery along a w-arrow. Note that the orientation of the portion of diagram containing the tail, needs to be specified

Figure 4.4: Surgery on a w-arrow.

to define the surgery move. In the case where a w-arrow contains some twist, surgery introduces a virtual crossing, as shown on the left-hand side of Figure 4.5. Moreover, if the edge of the w-arrow intersects the diagram D, or an edge of another w-arrow, then the surgery introduces virtual crossings as indicated on the right-hand side of Figure 4.5.

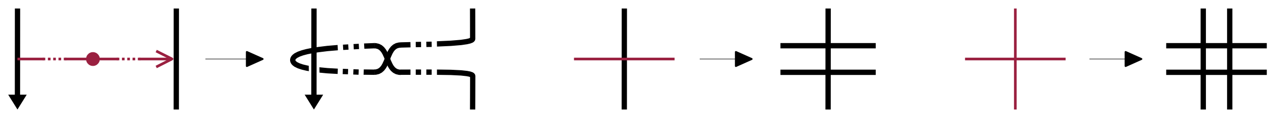


Figure 4.5: Surgeries near a twist and crossings.

Now if F contains some w-trees with degree higher than one, we first apply the *expanding rule* shown on Figure 4.6[1] at each trivalent vertex: this breaks up F into a union of w-arrows, on which we can perform surgery.

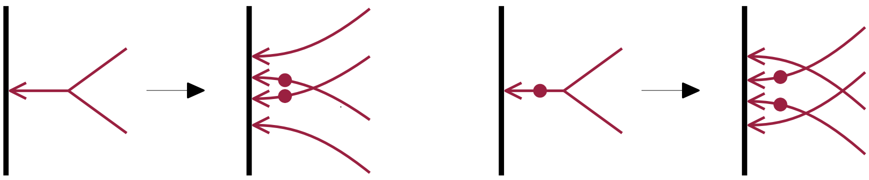


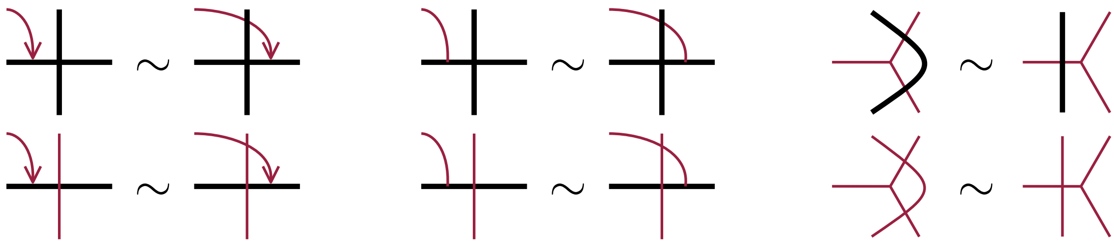
Figure 4.6: The expanding rule.

We describe in the following the homotopy arrow calculus. It refers to the set of operations on unions of welded tangles with some w-trees, which enable link-homotopic surgery results. These operations are developed in [MY19], and we summarize them in the next lemmas.

Lemma 4.1.8. *[MY19, Lemma 9.2] Surgery along a repeated w-tree does not change the homotopy class of a diagram.*

Proposition 4.1.9. *[MY19] We have the following homotopy equivalences.*

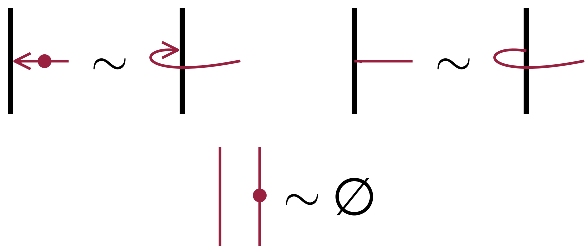
- *Arrow isotopy. Virtual Reidemeister moves involving edges of w-trees and/or strands of diagrams, together with the following local moves:*



- *Head/Tail Reversal. Changing the side of the strand from which an endpoint of a w-tree is attached has the following effect.*

- *Inverse. Two parallel copies of a w-tree which differ only by one twist, can be deleted up to homotopy.*

[1]Here and in the following figures, we use the diagrammatic convention adopted in [MY19, Convention 5.1].



Until the end of the section, T and S will denote w-trees for a given diagram D. As for clasper calculus, we use the notation $T \sim S$ to mean that $D_T \sim D_S$. The next proposition describes how to handle twists in the homotopy arrow calculus.

Proposition 4.1.10. *[MY19] We have the following homotopy equivalences (illustrated in Figure 4.7).*

(1) If T is obtained from S by removing two consecutive twists on an edge then $T \sim S$.

(2) If T is obtained from S by moving a twist across a crossing involving either an edge of a w-tree or a strand of a diagram then $T \sim S$.

(3) If T is obtained from S by moving a twist across a trivalent vertex then $T \sim S$.

(4) If T and S are identical outside a neighborhood of trivalent vertex, and if in this neighborhood T and S are as depicted in (4) from Figure 4.7, then $T \sim S$.

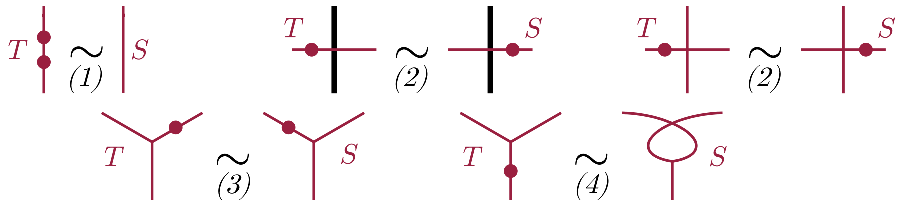


Figure 4.7: How to deal with twists in homotopy arrow calculus.

The following lemma describes how to exchange endpoints up to homotopy.

Lemma 4.1.11. *[MY19] We have the following homotopy equivalences (illustrated in Figure 4.8).*

(5) Tails exchange. If T and S have two adjacent tails and if $T' \cup S'$ is obtained from $T \cup S$ by exchanging these tails, then $T \cup S \sim T' \cup S'$.

(6) Heads exchange. If the heads of T and S are adjacent and if $T' \cup S'$ is obtained from $T \cup S$ by exchanging these heads as depicted in (6) Figure 4.8, then $T \cup S \sim T' \cup S' \cup \tilde{T}$, where $\tilde{T}$ is as shown in the figure.

(7) Head/Tail exchange. If the head of T is adjacent to a tail of S and if $T' \cup S'$ is obtained from $T \cup S$ by exchanging these endpoints as depicted in (7) from Figure 4.8, then $T \cup S \sim T' \cup S' \cup \tilde{T}$, where $\tilde{T}$ is as shown in the figure.

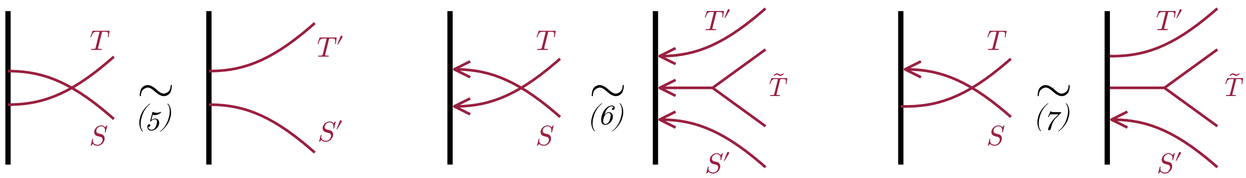


Figure 4.8: How to exchange endpoints in homotopy arrow calculus.

Finally, we have an arrow calculus version of the IHX relation.

Proposition 4.1.12. *[MY19, Lemma 7.14] Let T_I, T_H, T_X be three parallel copies of a given w-tree that coincide everywhere outside a disk, where they are as shown in Figure 4.9. Then $T_I \cup T_H \cup T_X \sim \varnothing$. We say that T_I, T_H and T_X* verify the IHX relation.

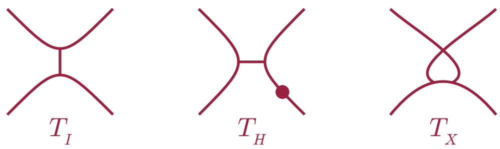


Figure 4.9: The IHX relation for w-trees.

4.2 Welded braids

This section is dedicated to *homotopy welded braids*. Our approach is similar to the one followed in Chapter 2 for classical braids. We will first define *comb-trees* which are the analogue of comb-claspers in the welded case. Then we study and improve presentations of welded braid groups using arrow calculus. We also show that the representation defined in Section 2.3 extends well in the welded context. Finally, we return to and fully solve the torsion problem addressed in Section 2.4.

4.2.1 Comb-trees

Let us take n fixed points, in the unit interval [0,1], denoted by $p_1 < p_2 < \cdots < p_n$.

Definition 4.2.1. *An n-*component **virtual braid diagram** *$\beta = (\beta_1, \ldots, \beta_n)$ is the image of a immersion*

$$(\beta_1, \ldots, \beta_n) : \bigsqcup_{i \leqslant n} [0,1] \to [0,1] \times [0,1]$$

such that, for some permutation of $\{1, \ldots, n\}$ associated to β, denoted $\pi(\beta)$, we have $\beta_i(0) = (p_i, 0)$ and $\beta_i(1) = (p_{\pi(\beta)(i)}, 1)$ for any i. We require the singularities to be a finite number of transverse double points, which are labeled either as classical crossings or as virtual crossings. Additionally, we require the immersion to be monotonic, which means that $\beta_i(t) \in [0,1] \times \{t\}$ for any $t \in [0, 1]$ and any i. We call the image of β_i the i-th component *of β. We say that a virtual braid diagram β is* **pure** *if its associated permutation $\pi(\beta)$ is the identity.*

The set of virtual braid diagrams up to welded isotopy (resp. homotopy), equipped with the stacking operation, forms a group: the **welded braid group** denoted by WB_n (resp. the **homotopy welded braid group**, denoted by hWB_n). Elements of hWB_n are called *homotopy welded braids*. The set of pure braids up to welded isotopy (resp. homotopy) forms a subgroup of WB_n (resp. hWB_n) denoted by WP_n (resp. hWP_n). Note that we do not require welded isotopy or homotopy to preserve the monotonic property during the deformation. As in the classical case (Remark 2.1.5), we will regard homotopy welded braid as welded tangles up to homotopy.

Proposition 4.2.2. *[ABMW17a, Theorem 3.10] Any welded string-link is link-homotopic to a pure welded braid, and if two pure welded braids are link-homotopic as string-links, then they also are as braids.*

We next introduce *comb-trees* and their associated notation. Let $I = (i_0, i_1, \ldots, i_l)$ be a sequence of nonrepeated indices in $\{1, \ldots, n\}$ such that $i_1 < i_j$ for any $2 \leqslant j \leqslant l$.

Definition 4.2.3. *The* **comb-trees** χ_I *and* χ_I^{-1}, *are the w–trees for the trivial n–braid diagram* $\mathbf{1}_n$, *shown in Figure 4.10. We say that* χ_I *is* **positive** *and that* χ_I^{-1} *is* **negative**.

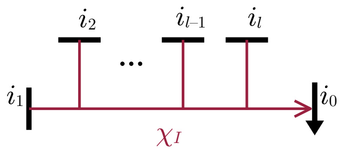


Figure 4.10: The positive and negative comb-trees χ_I and χ_I^{-1}.

In what follows, we blur the distinction between comb-trees and the result of their surgery up to homotopy. From this point of view, a comb-tree is a pure homotopy welded braid and the product $\chi_I\chi_{I'}$ of two comb-trees is the product $\mathbf{1}_{\chi_I}\mathbf{1}_{\chi_{I'}}$.

Example 4.2.4. *The two comb-trees* χ_I *and* χ_I^{-1} *are, up to homotopy, inverse to each other. Indeed, consider the product* $\chi_I\chi_I^{-1}$ *and apply an arrow isotopy to make the two comb-trees parallel. Then using the Inverse move from proposition 4.1.9 we delete the two w-trees. We illustrate this operation with the comb-trees* χ_{314} *and* χ_{314}^{-1} *in Figure 4.11.*

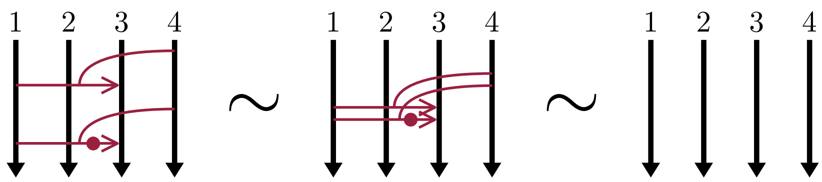


Figure 4.11: The product $\chi_{314}\chi_{314}^{-1}$ is trivial.

Lemma 4.2.5. *Let T be a w-tree of degree l for the trivial braid* $\mathbf{1}$ *with head on the i-th component, then* $\mathbf{1}_T$ *is link-homotopic to a product of comb-trees of degree l with head on the i-th component.*

Proof. First, we may use a Head Reversal from Proposition 4.1.9 to ensure that the orientation of the head and the strand that it intersects are arranged as in Figure 4.10. Then, we may apply the IHX relation of Lemma 4.1.12, and we may arrange the cyclic order at each trivalent vertex with Proposition 4.1.10 to get the shape depicted in Figure 4.10 for each individual comb-tree. Next, we may exchange endpoints using Lemma 4.1.11 to obtain the product arrangement; this creates w-trees with repeats which are trivial up to link-homotopy by Proposition 4.1.8. Finally, with Proposition 4.1.10, we move all twists to the edge incident to the head and cancel them pairwise. □

Definition 4.2.6. *We say that a pure homotopy welded braid $\beta \in hWP_n$ given by a product of comb-trees $\beta = \chi_{I_1}^{\pm 1}\chi_{I_2}^{\pm 1}\cdots\chi_{I_m}^{\pm 1}$ is :*

- stacked *if $\chi_{I_i} = \chi_{I_j}$ for some $i \leqslant j$ implies that $\chi_{I_i} = \chi_{I_k}$ for any $i \leqslant k \leqslant j$,*
- reduced *if it contains no redundant pair, i.e., two consecutive factors are not the inverse of each other.*

If β is reduced and stacked, then we can rewrite β as $\beta = \prod \chi_{I_i}^{\nu_i}$ for some integers ν_i and with $\chi_{I_i} \neq \chi_{I_j}$ for any $i \neq j$. Moreover, given any total order $\leqslant$ on the set of positive comb-trees, we say that a reduced and stacked writing is a **normal form** of β for this order *if $\chi_{I_i} \leqslant \chi_{I_j}$ for all $i \leqslant j$.*

Theorem 4.2.7. *Any pure homotopy welded braid $\beta \in hWP_n$ can be expressed in a normal form, for any order on the set of positive comb-trees.*

Proof. Note that the comb-trees χ_{ij} correspond to the usual pure welded braid group generators χ_{ij} shown in Figure 4.13 (see Remark 4.2.10). Thus it is clear that β is given by a product of degree one comb-trees.

Now we rearrange these comb-trees according to the given order with Lemma 4.1.11. This may introduce new w-trees of degree strictly higher than one, and by Lemma 4.2.5 we can freely assume that these are all comb-trees. Next we consider degree two comb-trees and we rearrange them according to the order. Again this introduces higher degree w-trees, which can all be assumed to be comb-trees according to Lemma 4.2.5. By iterating this process degree by degree, we eventually obtain the desired normal form. Indeed, the procedure terminates because w-trees of degree higher than n are trivial in hWP_n by Lemma 4.1.8. □

Remark 4.2.8. *This result is to be compared with Theorem 9.4 of [MY19], which uses a different notion of comb-tree, ordered according to the degree. Their method is based on the correspondence between comb-trees and Milnor numbers. In particular, their approach also proves the unicity of the normal form and the that Milnor numbers are complete invariants of pure homotopy welded braids. We will also prove the unicity of the normal form later in Corollary 4.2.35 using another method.*

Remark 4.2.9. *Note that this result could be adapted to the whole homotopy welded braid group. This would require extending the notion of normal form to all homotopy welded braids. This could be done by associating a homotopy welded braid with each permutation.*

4.2.2 Welded braid group presentations

In this section we use the usual Artin braid generators σ_i for $i \in \{1, \ldots, n-1\}$ and the usual virtual braid generators ρ_i for $i \in \{1, \ldots, n-1\}$ illustrated in Figure 4.12. We also use the usual pure welded braid generators

$$\chi_{ij} = \begin{cases} \rho_i\rho_{i+1}\cdots\rho_{j-2}\sigma_{j-1}\rho_{j-1}\rho_{j-2}\cdots\rho_{i+1}\rho_i & \text{if } 1 \leqslant i < j \leqslant n, \\ \rho_{i-1}\rho_{i-2}\cdots\rho_{j+1}\rho_j\sigma_j\rho_{j+1}\cdots\rho_{i-2}\rho_{i-1} & \text{if } 1 \leqslant j < i \leqslant n, \end{cases}$$

illustrated in Figure 4.13.

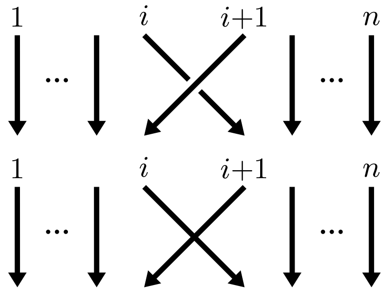


Figure 4.12: The welded generator σ_i and ρ_i.

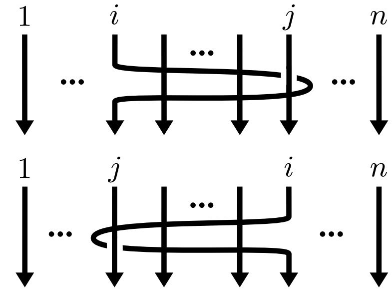


Figure 4.13: The pure welded generator χ_{ij}.

Remark 4.2.10. *Note that, the notation χ_{ij} is already used for degree one comb-trees from Section 4.2.1: this is because the pure welded braid generator χ_{ij} is the surgery result of the comb-tree χ_{ij} on the trivial braid.*

The (pure) homotopy welded braid group appears as a quotient of the (pure) welded braid group of which we recall a presentation from [Dam17] in Theorem 4.2.11. The end of the section consists in using the arrow calculus to describe some relations of the homotopy quotient in order to obtain a presentation for the (pure) homotopy welded braid group.

Theorem 4.2.11. *[Dam17, Corollary 3.15; Corollary 3.19.] A presentation[2] for the welded braid group is given by:*

$$WB_n = \left\langle \sigma_i,\rho_i \;\middle|\; \begin{array}{ll} [\sigma_i,\sigma_j] = 1 & \textit{if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \textit{for } 1 \leqslant i < n-1 \\ [\rho_i,\rho_j] = 1 & \textit{if } |i-j| > 1 \\ \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1} & \textit{for } 1 \leqslant i < n-1 \\ \rho_i^2 = 1 & \textit{for } 1 \leqslant i < n \\ [\sigma_i,\rho_j] = 1 & \textit{if } |i-j| > 1 \\ \sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1} & \textit{for } 1 \leqslant i < n-1 \\ \rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1} & \textit{for } 1 \leqslant i < n-1 \end{array} \right\rangle.$$

A presentation for the pure welded braid group is given by:

$$WP_n = \left\langle \chi_{ij} \;\middle|\; \begin{array}{ll} [\chi_{ij},\chi_{kl}] = 1 & \textit{if } \{i,j\} \cap \{k,l\} = \varnothing \\ [\chi_{ik},\chi_{jk}] = 1 & \textit{for any } i,j,k \\ [\chi_{ik}\chi_{jk},\chi_{ij}] = 1 & \textit{for any } i,j,k \end{array} \right\rangle.$$

[2]Here and in the following presentation, generators σ_i and ρ_i are indexed by integers $i \in \{1, \ldots, n-1\}$, and generators χ_{ij} are indexed by pairs of integers $i \neq j \in \{1, \ldots, n-1\}$.

In order to get a presentation for the pure homotopy welded braid group, let us first state a preparatory technical lemma.

Lemma 4.2.12. *For any $1 \leqslant i \neq k \leqslant n$ and any $\omega \in hWP_n$, the conjugate $\omega\chi_{ik}\omega^{-1}$ is obtained as the surgery on a product of nonrepeated w-trees for the trivial braid, all containing i and k in their support and having a tail on the k-th component.*

Proof. Firstly, if T_1 and T_2 are two w-trees for the trivial braid $\mathbf{1}$ such that $|\text{supp}(T_1) \cap \text{supp}(T_2)| \geqslant 2$ then their endpoints can be freely exchanged and T_1 commutes with T_2. Indeed, exchanging two tails is always possible according to move *(5)* from Lemma 4.1.11, and if one of the two endpoints is a head then by move *(6)* or *(7)* from Lemma 4.1.11 the exchange creates a new w-tree. However, thanks to the condition on the supports, this w-tree has repeats and is therefore trivial up to homotopy as shown by Lemma 4.1.8. Observe also that T_1 and T_2 commute if they have disjoint support, or if the endpoints on $\text{supp}(T_1) \cap \text{supp}(T_2)$ are all tails, by move *(5)* of Lemma 4.1.11.

The only remaining case is thus that of two w-trees T_1 and T_2 with $\text{supp}(T_1) \cap \text{supp}(T_2) = \{i\}$ for some i, and such that the i-th component contains the head of T_1 or T_2. Then commuting T_1 and T_2 is achieved by exchanging their endpoints lying on the i-th component. By doing so, as shown by move *(6)* or *(7)* from Lemma 4.1.11, this creates a new w-tree T_3 such that $\text{supp}(T_3) = \text{supp}(T_1) \cup \text{supp}(T_2)$ and $\text{roots}(T_3) = \text{roots}(T_1) \cup \text{roots}(T_2)$. This w-tree has then at least two endpoints in common with T_1 and T_2, thus it can be moved freely and we get $T_1T_2 = T_3T_2T_1 = T_2T_3T_1 = T_2T_1T_3$.

The observations above imply that, if F is a product of w-trees all containing i and k in their support and having a tail on the k-th component, then for any w-tree W the conjugate WFW^{-1} is again a product of w-trees all containing i and k in their support and having a tail on the k-th component.

Finally we express $\omega \in hWP_n$ as a product of w-trees $\omega = W_m \cdots W_0$. Then, since χ_{ik} contains i and k in its support and has a tail on the k-th component, the conjugate $F_1 := W_0\chi_{ik}W_0^{-1}$ is again a product of w-trees all containing i and k in their support and having a tail on the k-th component. Therefore, by taking successive conjugates $F_{k+1} := W_kF_kW_k^{-1}$, we eventually conclude the proof. □

The two following propositions give us relations in the pure homotopy welded braid group.

Proposition 4.2.13. *For any pairwise distinct $1 \leqslant i, j, k \leqslant n$, and any $\omega \in hWP_n$, χ_{jk} commutes with $\omega\chi_{ik}\omega^{-1}$.*

Proof. Let us denote by W a product of w-trees for the trivial braid $\mathbf{1}$ with surgery result $\omega\chi_{ik}\omega^{-1}$. To prove the proposition we consider the product $\chi_{jk}W$, and perform endpoints exchanges to move χ_{ij} down across W. To do so, we first slide the tail of χ_{jk} along the k-th component. Thanks to Lemma 4.2.12, all the factors of W have only a tail on the k-th component, so using move *(5)* from Lemma 4.1.11 we can achieve this sliding freely. Next, we slide the head of χ_{jk} along the j-th component. We use moves *(6)* and *(7)* from Lemma 4.1.11 to cross endpoints of w-trees in W that we encounter along the sliding. This creates w-trees with repeats (they intersect the k-th component in two points), which are trivial up to homotopy by Lemma 4.1.8. □

Proposition 4.2.14. *For any $1 \leqslant i \neq k \leqslant n$ and any $\omega \in hWP_n$, χ_{ki} commutes with $\omega\chi_{ik}\omega^{-1}$.*

Proof. Let us denote by W a product of w-trees for the trivial braid $\mathbf{1}$ with surgery result $\omega\chi_{ik}\omega^{-1}$. According to Lemma 4.2.12 all factors of W contains i and k in their support. In particular, when we use Lemma 4.1.11 to exchange endpoints of those factors with endpoints of χ_{ki}, we create w-trees with repeats, which are trivial up to homotopy by Lemma 4.1.8. □

Theorem 4.2.15. *We have the following presentation for the homotopy welded braid group.*

$$hWP_n = \left\langle \chi_{ij} \;\middle|\; \begin{array}{ll} [\chi_{ij},\chi_{kl}] = 1 & \text{if } \{i,j\} \cap \{k,l\} = \varnothing \\ [\chi_{ij},\chi_{jk}] = [\chi_{ik},\chi_{ij}] & \text{for any } i,\ j,\ k \\ \left[\chi_{jk},\omega\chi_{ik}\omega^{-1}\right] = \left[\chi_{ki},\omega\chi_{ik}\omega^{-1}\right] = 1 & \text{for any } i,\ j,\ k \text{ and any word } \omega \end{array} \right\rangle.$$

Proof. In [Dar23, Theorem 5.8.], a presentation for the pure homotopy welded braid group is given from the group presentation of Theorem 4.2.11, by adding relations of the form

$$\left[\chi_{jk},\omega\chi_{ik}\omega^{-1}\right] = \left[\chi_{ki},\omega\chi_{ik}\omega^{-1}\right] = 1,$$

for certain indices i, j, k, and for certain $\omega \in hWP_n$. Our observation is that, such relations are true for all indices i, j, k and all $\omega \in hWP_n$, as stated in Proposition 4.2.13 and Proposition 4.2.14. We conclude the proof, by showing that the relations $[\chi_{ik}\chi_{jk},\chi_{ij}] = 1$ and $[\chi_{ij},\chi_{jk}] = [\chi_{ik},\chi_{ij}]$ are equivalent in hWP_n:

$$\begin{array}{lrll} & [\chi_{ik}\chi_{jk},\chi_{ij}] &= 1, & \\ \Leftrightarrow & \chi_{ik}\chi_{jk}\chi_{ij}\chi_{jk}^{-1}\chi_{ik}^{-1}\chi_{ij}^{-1} &= 1, & \\ \Leftrightarrow & \chi_{ik}\chi_{jk}\chi_{ij}\chi_{ik}^{-1}\chi_{jk}^{-1}\chi_{ij}^{-1} &= 1, & \text{by commuting } \chi_{jk}^{-1} \text{ with } \chi_{ik}^{-1} \\ \Leftrightarrow & \chi_{ik}\chi_{jk}\chi_{ij}\chi_{ik}^{-1}\chi_{ij}^{-1}\chi_{ij}\chi_{jk}^{-1}\chi_{ij}^{-1} &= 1, & \\ \Leftrightarrow & \chi_{ik}\chi_{ij}\chi_{ik}^{-1}\chi_{ij}^{-1}\chi_{jk}\chi_{ij}\chi_{jk}^{-1}\chi_{ij}^{-1} &= 1, & \text{by commuting } \chi_{jk} \text{ with } \chi_{ij}\chi_{ik}^{-1}\chi_{ij}^{-1} \\ \Leftrightarrow & [\chi_{ik},\chi_{ij}][\chi_{jk},\chi_{ij}] &= 1, & \\ \Leftrightarrow & [\chi_{ij},\chi_{jk}] &= [\chi_{ik},\chi_{ij}]. & \end{array}$$

□

Corollary 4.2.16. *A presentation for the homotopy welded braid group hWB_n is given by adding the relations*

$$\left[\chi_{jk},\omega\chi_{ik}\omega^{-1}\right] = \left[\chi_{ki},\omega\chi_{ik}\omega^{-1}\right] = 1,$$

for any i, j, k, and any $\omega \in hWP_n$, to the presentation of the welded braid group in Theorem 4.2.11.

Proof. The proof follows from Theorem 4.2.15, and the fact that, if a welded braid is trivial up to homotopy then it belongs to the pure welded braid group hWP_n. □

Remark 4.2.17. *We give here, in Theorem 4.2.15 and Corollary 4.2.16, infinite presentations. However, as in the classical framework (see Remarks 2.2.3 and 2.2.7), we can reduce them to finite presentations using Proposition 1.2.5 and Corollary 1.2.6.*

4.2.3 A linear faithful representation of the homotopy welded braid group

In this section, we extend the construction of the representation γ of Section 2.3 to the framework of the homotopy welded braid group. As a result, the construction of this section is very similar to that of Section 2.3. To avoid being too redundant, we will go a little faster and omit some proofs when we consider them too identical to those of Section 2.3.

4.2.3.1 Algebraic definition

Let us first recall some algebraic ingredients from Section 1.2. First, we need the reduced free group $\mathcal{R}F_n$ from Definition 1.2.1. It is the quotient of the free group in which the generators x_i commute with their conjugates. We showed in Theorem 1.2.10 that any element $\omega \in \mathcal{R}F_n$ has a unique normal form, i.e., there exists a unique ordered set of integers $\{e_1, \ldots, e_m\}$ associated to the ordered family of commutators $\mathcal{F} = \{[\alpha_1], [\alpha_2], \ldots, [\alpha_m]\}$ such that we have a unique writing

$$\omega = [\alpha_1]^{e_1}[\alpha_2]^{e_2} \cdots [\alpha_m]^{e_m}.$$

Recall that the elements $[\alpha] \in \mathcal{F}$ are given for a sequence of indices $\alpha = (i_1,i_2,\cdots,i_l)$, by the following expression:

$$[\alpha] := [[\cdots[[x_{i_1},x_{i_2}],x_{i_3}],\cdots,x_{i_{l-1}}],x_{i_l}] \in \mathcal{R}F_n.$$

Finally, in Definition 1.2.13, we defined the **Z**-module $\mathcal{V}$ generated by the formal commutators $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ associated to the family $\mathcal{F}$. We also defined the linearization map $\phi : \mathcal{R}F_n \to \mathcal{V}$ given on an ordered normal form by:

$$\phi([\alpha_1]^{e_1}[\alpha_2]^{e_2} \cdots [\alpha_m]^{e_m}) = e_1\alpha_1 + e_2\alpha_2 + \cdots + e_m\alpha_m.$$

In order to define a linear representation of the homotopy welded braid group, we need the *homotopy welded Artin representation.*

Definition 4.2.18. *We call* **welded Artin representation** *the homomorphism denoted by* $\zeta : WB_n \to Aut(F_n)$ *and defined as follows:*

$$\zeta(\sigma_i) : \begin{cases} x_i & \mapsto & x_{i+1}, \\ x_{i+1} & \mapsto & x_{i+1}^{-1}x_ix_{i+1}, \\ x_k & \mapsto & x_k & \text{if } k \notin \{i,\, i+1\}, \end{cases}$$

and,

$$\zeta(\rho_i) : \begin{cases} x_i & \mapsto & x_{i+1}, \\ x_{i+1} & \mapsto & x_i, \\ x_k & \mapsto & x_k & \text{if } k \notin \{i,\, i+1\}. \end{cases}$$

Similarly, the homomorphism

$$\zeta_h : hWB_n \to Aut(\mathcal{R}F_n)$$

defined by the same expressions is called the **homotopy welded Artin representation**.

The fact that the homotopy welded Artin representation is well-defined is discussed in [Dah62]; see also [Wat72, FRR97]. The fact that the homotopic version of this representation, it is shown in [ABMW17a, Section 4.4.1] that it is well-defined.

Theorem 4.2.19. *Let* $GL(\mathcal{V})$ *be the general linear group of the* **Z**-*module* $\mathcal{V}$. *The map*

$$\gamma_W : hWB_n \to GL(\mathcal{V})$$

defined for $\beta \in hWB_n$ *and* $[\alpha] \in \mathcal{F}$ *by* $\gamma_W(\beta)(\alpha) = \phi \circ \zeta_h(\beta)([\alpha])$ *is a well-defined homomorphism. Moreover,* γ_W *does not depend on the chosen order on* $\mathcal{F}$.

The proof of Theorem 4.2.19 is strictly similar to that of Proposition 2.3.2. It is based on Lemma 4.2.20, which is the welded analogue of Lemma 2.3.1, and which is proved in the same way.

Lemma 4.2.20. *Let us denote by N_j the subgroup normally generated by x_j in $\mathcal{R}F_n$ for $j \in \{1, \ldots, n\}$. Let $\beta \in hWB_n$ be a homotopy welded braid with associated permutation $\pi(\beta)$, and let $C \in N_j$ be a commutator. If the product $[\alpha_1]^{e_1} \cdots [\alpha_m]^{e_m}$ is the normal form of $\zeta_h(\beta)(C)$ then we have that $e_i = 0$ if $[\alpha_i] \notin N_{\pi^{-1}(\beta)(j)}$.*

Remark 4.2.21. *We mention that, like the representation γ, the homomorphism γ_W is injective. This can be shown using the injectivity of ϕ and ζ_h (see [ABMW17a, Proposition 2.33]). However, in the next Section 4.2.3.4, we will give another proof of this result using arrow calculus. This is stated in Theorem 4.2.34, which in turn reproves the injectivity of ζ_h. Furthermore, our approach by arrow calculus will allow explicit computations of the representation in Section 4.2.3.3.*

Remark 4.2.22. *The representation theory of welded braid groups is a new and rich field of research: so far, few other representations are known, and the focus is mainly on extending Burau's representation, see for instance [KMRW17, PS22, DMR23].*

In the following proposition, we prove the well-know fact that the natural inclusion of hB_n in hWB_n is injective.

Proposition 4.2.23. *The homotopy braid group hB_n injects into the homotopy welded braid group hWB_n as follows:*

$$\begin{array}{cccc} \iota \ : & hB_n & \to & hWB_n \\ & \sigma_i & \mapsto & \sigma_i \end{array}$$

Proof. Let us take $\beta \in hB_n$ in the kernel of ι, then $\gamma_W \circ \iota(\beta) = \gamma_W(\mathbf{1})$. Moreover, we see from the definition that the image $\gamma(\sigma_i)$ of $\sigma_i \in hB_n$ by the representation γ defined in Section 2.3, coincides with the image $\gamma_W(\sigma_i)$ of $\sigma_i \in hWB_n$ for any $1 \leqslant i \leqslant n$. In particular we obtain the equality $\gamma_W \circ \iota(\beta) = \gamma(\beta) = \mathrm{Id}$. Finally, using the injectivity of γ from Theorem 2.3.11, we get that $\beta = \mathbf{1}$ and the proof is complete. □

Remark 4.2.24. *From this proof, we can freely regard the representation γ of Section 2.3, as the restriction of γ_W to hB_n, seen as a subgroup of hWB_n.*

4.2.3.2 Arrow interpretation

We first give an interpretation of the welded Artin (resp. homotopy welded Artin) representation in terms of arrow calculus. As in the classical case, we first add a new strand to the right of the braid and we label it by '∞'. Then we give in the following lemma a diagrammatic interpretation of the free group F_n (resp. reduced free group $\mathcal{R}F_n$) on which WB_n (resp. hWB_n) acts. To do so we introduce the pure generator $\chi_{\infty,i}$ for $1 \leqslant i \leqslant n$ shown in Figure 4.14. This generator $\chi_{\infty,i}$ can be reinterpreted in terms of arrows as depicted in the same figure. There and in subsequent figures, we simply represent a small part of the ∞ component on which the arrow head is located.

Lemma 4.2.25. *The family $\{\chi_{\infty,i} = \rho_n\rho_{n-1}\cdots\rho_{i+1}\rho_i\sigma_i\rho_{i+1}\cdots\rho_{n-1}\rho_n\}_{1\leqslant i\leqslant n}$ seen as pure welded braids in WB_{n+1} (resp. homotopy welded braid in hWB_{n+1}) generate a free group (resp. reduced free group).*

Figure 4.14: The pure generator $\chi_{\infty,i}$ and its arrow interpretation.

Proof. We only provide the proof in the homotopy setting, since this is the version that we will use afterwards. Note, however, that the proof is very similar in the non-homotopic case. We first use the homotopy welded Artin representation to reinterpret the $\chi_{\infty,i}$ as automorphisms of the reduced free group generated by $x_1, \ldots, x_n, x_\infty$ as follows:

$$\zeta_h(\chi_{\infty,i}) = \begin{cases} x_\infty & \mapsto & x_i^{-1} x_\infty x_i, \\ x_k & \mapsto & x_k & \text{if } k \leqslant n. \end{cases}$$

In particular, for any element $\chi_{\infty,i_1} \cdots \chi_{\infty,i_m} \in \langle \chi_{\infty,i} \rangle_{1 \leqslant i \leqslant n}$ we have:

$$\zeta(\chi_{\infty,i_1} \cdots \chi_{\infty,i_m})(x_\infty) = x_{i_1}^{-1} \cdots x_{i_m}^{-1} x_\infty x_{i_1} \cdots x_{i_m},$$

with $x_{i_k} \neq x_\infty \in \mathcal{R}F_{n+1}$ for any $1 \leqslant k \leqslant m$. Therefore, if a relation $\chi_{\infty,i_1} \cdots \chi_{\infty,i_m} = 1$ holds in $\langle \chi_{\infty,i} \rangle_{1 \leqslant i \leqslant n}$, then the relation $x_{i_1} \cdots x_{i_m} = 1$ must also hold in $\mathcal{R}F_{n+1}$. However, $\mathcal{R}F_{n+1}$ only admits reduced-type relations (i.e., of the form $[x_i, \lambda x_i \lambda^{-1}] = 1$ for any i and any $\lambda \in \mathcal{R}F_n$), thus the only possible relations in $\langle \chi_{\infty,i} \rangle_{1 \leqslant i \leqslant n}$ must be of reduced-type as well. But we have seen in Theorem 4.2.15 that the generators $\chi_{\infty,i}$ indeed satisfy all reduced-type relations $[\chi_{\infty,i}, \omega \chi_{\infty,i} \omega^{-1}] = 1$ for any $1 \leqslant i \leqslant n$ and any $\omega \in \langle \chi_{\infty,i} \rangle_{1 \leqslant i \leqslant n}$. □

In this context, the automorphism $\zeta(\beta)$ (resp. $\zeta_h(\beta)$) associated to an element β in WB_n (resp. hWB_n) is given on a generator $\chi_{\infty,i}$ in F_n (resp. in $\mathcal{R}F_n$) by considering the conjugation $\beta \chi_{\infty,i} \beta^{-1}$ illustrated in Figure 4.15. Then we re-express this element as a product of w-trees with heads on the

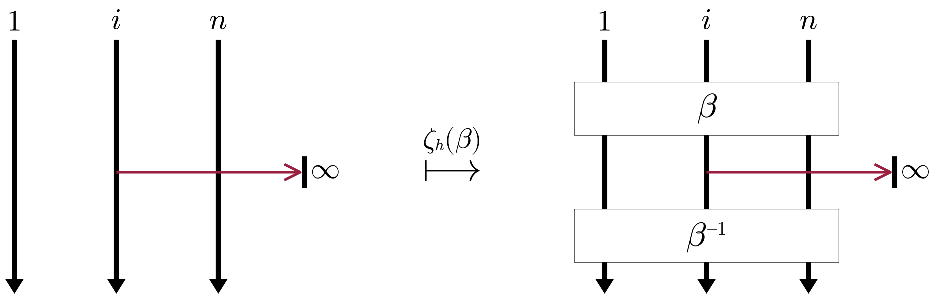


Figure 4.15: Arrow interpretation of the welded Artin representation.

∞-strand, which we are able to reinterpret as elements of F_n or $\mathcal{R}F_n$. This fact is explicitly stated in the following lemma in the homotopic framework.

Lemma 4.2.26. *We have the following equalities in hWB_{n+1}:*

$$\rho_i \chi_{\infty,k} \rho_i = \begin{cases} \chi_{\infty,i+1} & \text{if } k = i, \\ \chi_{\infty,i} & \text{if } k = i+1, \\ \chi_{\infty,k} & \text{otherwise,} \end{cases}$$

and,

$$\sigma_i\chi_{\infty,k}\sigma_i^{-1} = \begin{cases} \chi_{\infty,i+1} & \text{if } k = i, \\ \chi_{\infty,i+1}^{-1}\chi_{\infty,i}\chi_{\infty,i+1} & \text{if } k = i+1, \\ \chi_{\infty,k} & \text{otherwise.} \end{cases}$$

Proof. We compute $\rho_i\chi_{\infty,k}\rho_i$ using arrow calculus. If $k \notin \{i,i+1\}$, the equality is clear since $\chi_{\infty,k}$ commutes with ρ_i up to virtual isotopy. If $k = i$ (resp. $k = i+1$) we slide the tail of $\chi_{\infty,i}$ (resp. $\chi_{\infty,i+1}$) through ρ_i, obtaining $\chi_{\infty,i+1}$ (resp. $\chi_{\infty,i}$), and then simplify the two virtual associated with ρ_i^2.
Next, we turn our attention to the classical generators σ_i, and we compute $\sigma_i\chi_{\infty,k}\sigma_i^{-1}$. Again, when $k \notin \{i,i+1\}$ we have that $\chi_{\infty,k}$ commutes with σ_i and the equality is clear. If $k = i$, we rewrite the conjugate as

$$\sigma_i\chi_{\infty,i}\sigma_i^{-1} = \sigma_i\rho_i^2\chi_{\infty,i}\rho_i^2\sigma_i^{-1} = \chi_{i,i+1}\chi_{\infty,i+1}\chi_{i,i+1}^{-1},$$

where the second equality use the equality proved just above and the following $\sigma_i\rho_i = \chi_{i,i+1}$. Thanks to the tails exchange move from Lemma 4.1.11, $\chi_{i,i+1}$ and $\chi_{\infty,i+1}$ commute and we have the desired equality. For the last case, if $k = i+1$, applying the same trick transforms the conjugate into:

$$\sigma_i\chi_{\infty,i+1}\sigma_i^{-1} = \chi_{i,i+1}\chi_{\infty,i}\chi_{i,i+1}^{-1}.$$

This yields a new conjugate, illustrated on the left side of Figure 4.16. Finally, we conclude using a slide move from [MY19, Section 4.3] to transform this conjugate into $\chi_{\infty,i+1}^{-1}\chi_{\infty,i}\chi_{\infty,i+1}$, as depicted on the right-hand side of Figure 4.16. □

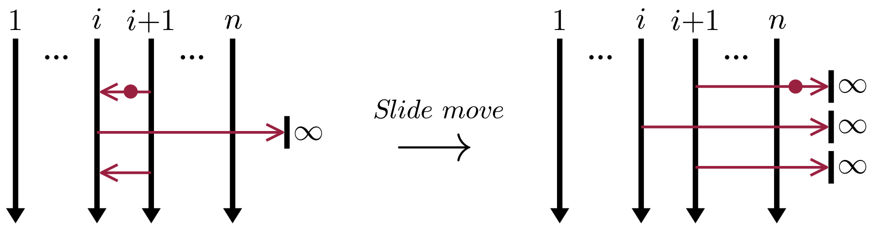


Figure 4.16: Slide move between $\chi_{i,i+1}\chi_{\infty,i}\chi_{i,i+1}^{-1}$ and $\chi_{\infty,i+1}^{-1}\chi_{\infty,i}\chi_{\infty,i+1}$.

Therefore, as in the classical case, we have an explicit 3-steps procedure to compute $\gamma_W(\beta)(\alpha)$ for any $\beta \in hWB_n$ and any $\alpha \in \mathcal{V}$:

Step 1 Consider the conjugate of the w-tree $\chi_{\infty,\alpha}$ by the braid β (see Figure 4.15).

Step 2 Use arrow calculus to re-express this conjugate as an ordered union of comb-trees of the form $\chi_{\infty,\alpha'}$ (the order comes from the order on $\mathcal{F}$).

Step 3 The number of parallel copies of a given comb-tree in this product is the coefficient of the associated commutator in $\gamma_W(\beta)(\alpha)$.

In the proof of Theorem 4.2.28 below, we will use explicitly this procedure to compute the representation.

Let us give first, in the following proposition, a correspondence between the family of commutators $\mathcal{F}$ and comb-trees of the form $\chi_{\infty,I}$.

Proposition 4.2.27. *Let $i_1, i_2, \ldots, i_l$ be a sequence of nonrepeated indices such that $i_1 < i_j$ for any $j \leqslant l$. We have the following relation up to homotopy:*

$$\chi_{\infty,i_1,\ldots,i_l} \sim [\chi_{\infty,i_1,\ldots,i_{l-1}}, \chi_{\infty,i_l}] = \chi_{\infty,i_1,\ldots,i_{l-1}} \cdot \chi_{\infty,i_l} \cdot (\chi_{\infty,i_1,\ldots,i_{l-1}})^{-1} \cdot (\chi_{\infty,i_l})^{-1}.$$

For example in Figure 4.17 we illustrate the equivalence $\chi_{\infty 1324} \sim [\chi_{\infty 132}, \chi_{\infty 4}]$.

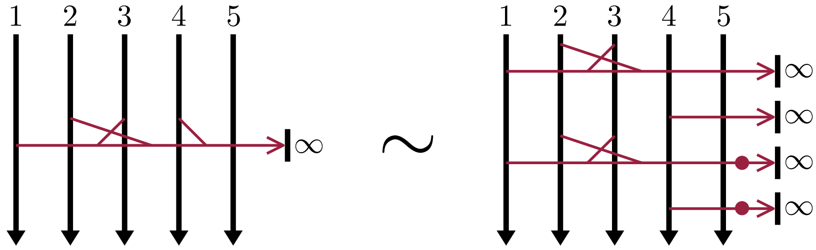


Figure 4.17: The w-tree $\chi_{\infty 1324}$ is link-homotopic to the commutator $[\chi_{\infty 132}, \chi_{\infty 4}]$.

Proof. Consider the product of w-trees $\chi_{\infty,i_1,\ldots,i_{l-1}} \cdot \chi_{\infty,i_l} \cdot (\chi_{\infty,i_1,\ldots,i_{l-1}})^{-1} \cdot (\chi_{\infty,i_l})^{-1}$ (as for example on the right-hand side of Figure 4.17). First, we use move *(6)* from Lemma 4.1.11 to exchange the heads of $\chi_{\infty,i_1,\ldots,i_{l-1}}$ and χ_{∞,i_l}; this move creates an extra w-tree, which is exactly $\chi_{\infty,i_1,\ldots,i_l}$. Now using arrow isotopies and the inverse move from Proposition 4.1.9 we get:

$$\chi_{\infty,i_1,\ldots,i_l} \cdot \chi_{\infty,i_l} \cdot \chi_{\infty,i_1,\ldots,i_{l-1}} \cdot (\chi_{\infty,i_1,\ldots,i_{l-1}})^{-1} \cdot (\chi_{\infty,i_l})^{-1} \sim \chi_{\infty,i_1,\ldots,i_l}.$$

□

By using this proposition iteratively, we obtain a correspondence between the commutators $[\alpha] \in \mathcal{F}$ (or $\alpha \in \mathcal{V}$) and the w-trees $\chi_{\infty,\alpha}$. For example the homotopy equivalence

$$\chi_{\infty 1324} \sim [[[\chi_{\infty 1}, \chi_{\infty 3}], \chi_{\infty 2}], \chi_{\infty 4}]$$

corresponds to $[1324] = [[[x_1,x_3],x_2],x_4]$ in $\mathcal{R}F_n$.

4.2.3.3 Explicit Computations

In Theorem 2.3.5, we computed the representation γ on the Artin generators σ_i; This readily provides the computation of $\gamma_W(\sigma_i)$ by Remark 4.2.24. In order to describe the representation γ_W, it thus remains to compute its image on the virtual generators ρ_i. This is done in the next theorem using the above procedure. As in Theorem 2.3.5, the images of a commutator $(i_1,i_2,\cdots,i_l) := \phi([i_1,i_2,\cdots,i_l]) \in \mathcal{V}$ by the maps $\gamma_W(\sigma_i)$ and $\gamma_W(\rho_i)$, depend on the position of the indices i and $i+1$ in the sequence $i_1, i_2, \ldots, i_l$.

Theorem 4.2.28. *For suitable sequences I, J, K in $\{1, \ldots, n\} \backslash \{i, i+1\}$, $I \neq \varnothing$, we have:*

$$\gamma_W(\sigma_i) : \begin{cases} (I) & \mapsto \quad (I) & (a) \\ (J,i,K) & \mapsto \quad (J,i+1,K) & (b) \\ (i+1,K) & \mapsto \quad (i,K) + (i,i+1,K) & (c) \\ (I,i+1,K) & \mapsto \quad (I,i,K) + (I,i,i+1,K) - (I,i+1,i,K) & (d) \\ (I,i,J,i+1,K) & \mapsto \quad (I,i+1,J,i,K) & (e) \\ (I,i+1,J,i,K) & \mapsto \quad (I,i,J,i+1,K) & (f) \\ (i,J,i+1,K) & \mapsto \quad \sum_{J' \subseteq J} (-1)^{|J'|+1} (i,\overline{J'},i+1,J \backslash J',K) & (g) \end{cases}$$

and

$$\gamma_W(\rho_i):\begin{cases} (I) & \mapsto & (I) & (h)\\ (J,i,K) & \mapsto & (J,i+1,K) & (i)\\ (J,i+1,K) & \mapsto & (J,i,K) & (j)\\ (I,i,J,i+1,K) & \mapsto & (I,i+1,J,i,K) & (k)\\ (I,i+1,J,i,K) & \mapsto & (I,i,J,i+1,K) & (l)\\ (i,J,i+1,K) & \mapsto & \sum_{J'\subseteq J}(-1)^{|J'|+1}(i,\overline{J'},i+1,J\backslash J',K) & (m)\end{cases}$$

where in (g) and (m), the sum is over all (possibly empty) subsequences J' of J, and $\overline{J'}$ denotes the sequence obtained from J' by reversing the order of its elements, see Example 4.2.29.

Example 4.2.29. *If $J=(j_1,\ j_2,\ j_3)$ and $K=\varnothing$ in (g) or (m), then $\gamma_W(\sigma_i)$ and $\gamma_W(\rho_i)$ both map $(i,J,i+1)$ to :*

$$-(i,i+1,j_1,j_2,j_3)+(i,j_1,i+1,j_2,j_3)+(i,j_2,i+1,j_1,j_3)+(i,j_3,i+1,j_1,j_2)$$

$$-(i,j_2,j_1,i+1,j_3)-(i,j_3,j_1,i+1,j_2)-(i,j_3,j_2,i+1,j_1)+(i,j_3,j_2,j_1,i+1).$$

The proof below explains how this follows from the IHX relations of Figure 4.20.

Proof of Theorem 4.2.28. As already observed, the former half of the statement, expressing $\gamma_W(\sigma_i)$, readily follows from Theorem 2.3.5. Hence we focus here on computing $\gamma_W(\rho_i)$. Following the procedure given above, we consider the conjugate $\rho_i\chi_{\infty,\alpha}\rho_i^{-1}$ and apply arrow calculus to turn it into a union of w-trees with heads on the ∞-th component.

For (h) it is clear that $\chi_{\infty,I}$ commutes with ρ_i by arrow isotopy, since $i,\ i+1\notin\mathrm{supp}(\chi_{\infty,I})$. The computations of (i), (j), (k) and (l) are given by an isotopy interchanging the i-th and $i+1$-th component, as shown for example in Figure 4.18 in the case (i).

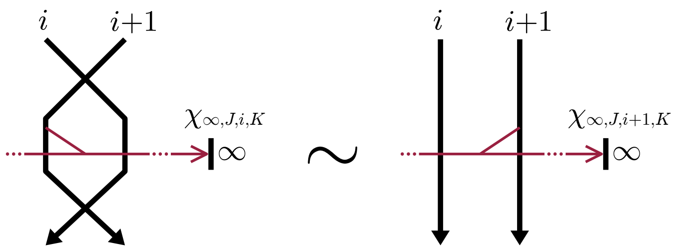


Figure 4.18: Computation of (i).

For (m), the first step is illustrated in Figure 4.19: we apply the previous isotopy followed by move *(4)* from Proposition 4.1.10, turning $\rho_i\chi_{\infty,i,J,i+1,K}\rho_i^{-1}$ into a new w-tree, which is not a comb-tree. In a second step, we use the IHX relation from Proposition 4.1.12 repeatedly to turn this new w-tree into a product of comb-trees. This is illustrated in Figure 4.20 where $J=(j_1,j_2,j_3)$, as in example 4.2.29; we conclude by simplifying the twists with Proposition 4.1.10 □

Example 4.2.30. *We illustrate Theorem 4.2.28 on the 3-component homotopy welded braid group hWB_3. To do so, we set* (1), (2), (3), (12), (13), (23), (123), (132) *to be the generators of $\mathcal{V}$, with the order of Definition 1.2.8. We already computed in Example 2.3.7 the automorphisms $\gamma(\sigma_1)$ and*

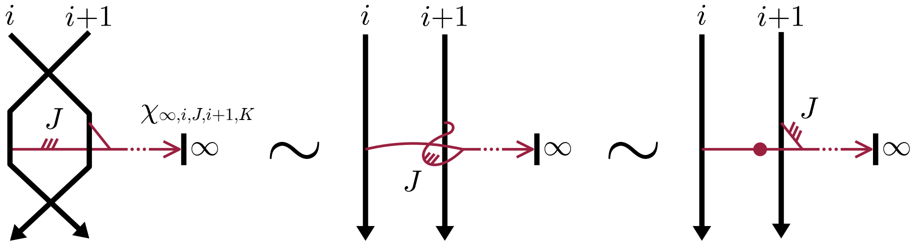


Figure 4.19: Turning $\rho_i \chi_{\infty,i,J,i+1,K} \rho_i^{-1}$ into a new w-tree.

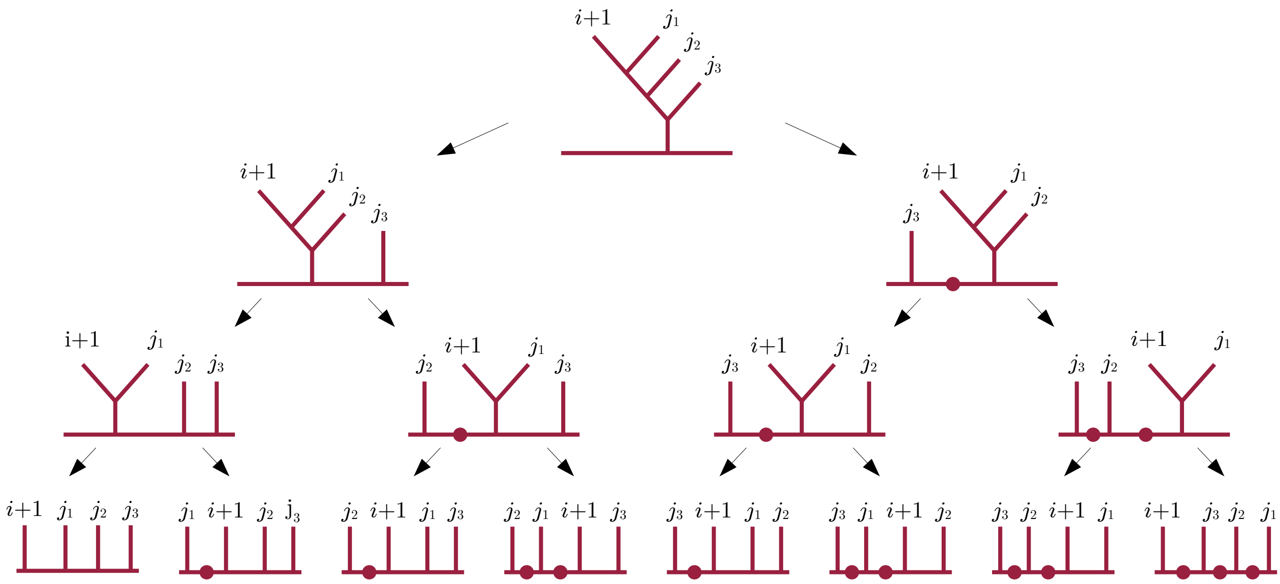


Figure 4.20: Iterated IHX relations.

$\gamma(\sigma_2)$, which coincide with $\gamma_W(\sigma_1)$ and $\gamma_W(\sigma_2)$ as mentioned by Remark 4.2.24. We compute here γ_W on the virtual generators ρ_1, ρ_2:

$$\begin{array}{ll}
\gamma_W(\rho_1)(1) = (2), & \gamma_W(\rho_2)(1) = (1), \\
\gamma_W(\rho_1)(2) = (1), & \gamma_W(\rho_2)(2) = (3), \\
\gamma_W(\rho_1)(3) = (3), & \gamma_W(\rho_2)(3) = (2), \\
\gamma_W(\rho_1)(12) = -(12), & \gamma_W(\rho_2)(12) = (13), \\
\gamma_W(\rho_1)(13) = (23), & \gamma_W(\rho_2)(13) = (12), \\
\gamma_W(\rho_1)(23) = (13), & \gamma_W(\rho_2)(23) = -(23), \\
\gamma_W(\rho_1)(123) = -(123), & \gamma_W(\rho_2)(123) = (132), \\
\gamma_W(\rho_1)(132) = -(123) + (132), & \gamma_W(\rho_2)(132) = (123).
\end{array}$$

This gives us the following matrices:

$$\gamma_W(\rho_1) = \left(\begin{array}{ccc|ccc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \quad \gamma_W(\rho_2) = \left(\begin{array}{ccc|ccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

Given the similarities between these matrices and those for $\gamma(\sigma_i) = \gamma_W(\sigma_i)$ given in example 4.2.30, it is not surprising to have the following analogue of Proposition 2.3.8 and Remark 2.3.9 in the welded case. The proof are omited since they are strictly similar to those given in Section 2.3.

Proposition 4.2.31. *For $\beta \in hWB_n$ a homotopy welded braid, the matrix associated to $\gamma_W(\beta)$ in the basis of $\mathcal{V}$, endowed with the order resulting from Definition 1.2.8, is given by a lower triangular block matrix of the following form:*

$$\begin{pmatrix}
B_{1,1} & 0 & \cdots & 0 \\
B_{2,1} & B_{2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
B_{n,1} & B_{n,2} & \cdots & B_{n,n}
\end{pmatrix}$$

where $B_{i,i}$ is a finite order matrix of size $rk(\mathcal{V}_i) = \sum_{i-1}^{n-1} \frac{k!}{(k-i+1)!}$ which is the identity when β is pure. Moreover, $B_{1,1}$ is the matrix of the left action by permutation $k \mapsto \pi^{-1}(\beta)(k)$, and $B_{2,2}$ is the matrix of the left action on the set $\{(k, j)\}_{k<j}$ given by:

$$(k, j) \mapsto \begin{cases} \left(\pi^{-1}(\beta)(k), \pi^{-1}(\beta)(j)\right) & \textit{if } \pi^{-1}(\beta)(k) < \pi^{-1}(\beta)(j), \\ -\left(\pi^{-1}(\beta)(j), \pi^{-1}(\beta)(k)\right) & \textit{if } \pi^{-1}(\beta)(j) < \pi^{-1}(\beta)(k). \end{cases}$$

Remark 4.2.32. *By the same argument as in Remark 2.3.9, the image $\gamma_W(\beta)(k)$ on all weight 1 commutators (k), is encoded in the blocks $B_{i,1}$ given in the first n columns, and these blocks thus completely determine $\gamma_W(\beta)$. Additionally, the blocks Bi,i are entirely determined by the permutation $\pi(\beta)$ associated with the braid $\beta \in hWB_n$.*

4.2.3.4 Injectivity

We conclude by showing the injectivity of the representation γ_W and the unicity of the normal form. As a preparatory step, let us first compute the image by γ_W of a braid given by the surgery result of a comb-tree.

Lemma 4.2.33. *Let $i_0 \in \{1, \ldots, n\}$ and let $J = (i_1, i_2, \ldots, i_k)$ be a sequence of non-repeated indices in $\{1, \cdots, n\}\backslash\{i_0\}$, such that $i_1 < i_l$ for all $l \leqslant k$. Let also i be any index in $\{1, \ldots, n\}$. The image of the comb-tree $\chi_{i_0 J}$ by the representation γ_W, applied to the commutator $(i) \in \mathcal{V}$, is given by the following:*

$$\gamma_W(\chi_{i_0 J})(i) = \begin{cases} (i) & \text{if } i_0 \neq i, \\ (i_0) - (J, i_0) & \text{if } i_0 = i \text{ and } i_1 < i_0, \\ (i_0) + (i_0, J) + S & \text{if } i_0 = i \text{ and } i_0 < i_1, \end{cases}$$

with S a linear combination of commutators in $\mathcal{V}$ of the form $\left(i_0, i_{\tau(1)}, \cdots, i_{\tau(k)}\right)$, for some permutations τ such that $i_{\tau(1)} \neq i_1 = \min(J)$.

In Figure 4.21 we illustrate the relation $\gamma(\chi_{4135})(4) = (4) - (1354)$.

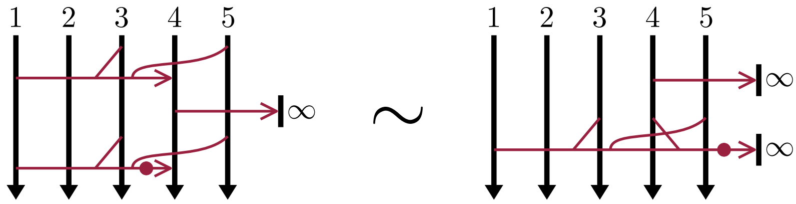


Figure 4.21: The relation $\gamma(\chi_{4135})(4) = (4) - (1354)$.

In Figure 4.22 we illustrate the relation $\gamma(\chi_{245})(2) = (2) + (245) - (254)$.

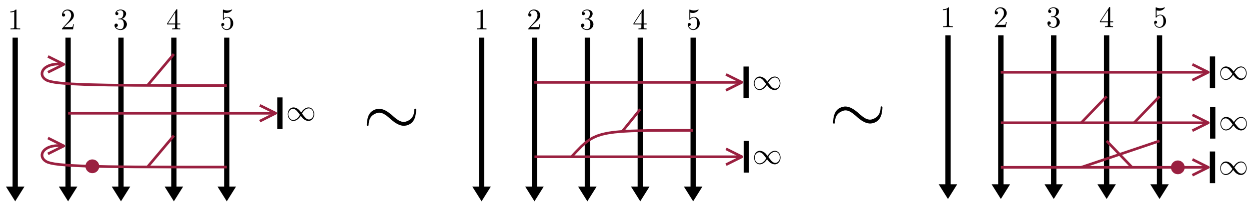


Figure 4.22: The relation $\gamma(\chi_{245})(2) = (2) + (245) - (254)$.

Proof. Following the 3-steps procedure of section 4.2.3.2, we consider the product $\chi_{i_0 J}\chi_{\infty i_j}\chi_{i_0 J}^{-1}$ and re-express it with only comb-trees whose head is on the ∞-th component. To do this, we want to commute $\chi_{i_0 J}$ and $\chi_{\infty i_j}$, then simplify $\chi_{i_0 J}$ and $\chi_{i_0 J}^{-1}$ using the inverse move from Proposition 4.1.9. To commute $\chi_{i_0 J}$ and $\chi_{\infty i_j}$, we may need Lemma 4.1.11 to exchange the tail of $\chi_{i_0 J}$ with an endpoint of $\chi_{i_0 J}$. This can be achieved for free if the head of $\chi_{i_0 J}$ is not on the i_k-th component, i.e., if $i_0 \neq i_k$. Otherwise, we apply a Head/Tail exchange (move *(7)*), which creates an extra w-tree (see Figures 4.21 and 4.22 for examples). If $i_1 < i_0$, this new w-tree is exactly the comb-tree $\chi_{\infty J i_0}^{-1}$. If $i_0 < i_1$, we have to apply the IHX relation from Proposition 4.1.12 repeatedly to turn it into a product of comb-trees of the form $\chi_{\infty i_0 i_{\tau(1)}\cdots i_{\tau(k)}}$, for some permutation τ. Note that, in the process, the only factor $\chi_{\infty i_0 i_{\tau(1)}\cdots i_{\tau(k)}}$ satisfying $\tau(1) = 1$ is the comb-tree $\chi_{\infty i_0 J}$.[3] □

Theorem 4.2.34. *The representation $\gamma_W : hWB_n \to GL(\mathcal{V})$ is injective.*

Proof. We take $\beta \in \ker(\gamma_W)$, which is pure according to Proposition 4.2.31; otherwise the block $B_{1,1}$ is not the identity. Then, we consider a normal form for β using Theorem 4.2.7:

$$\beta = \prod \chi_{iJ}^{\nu_{iJ}}.$$

The rest of the proof follows the same strategy as in Theorem 2.3.11. However, this time we use another sub-representation adapted to the welded case. Consider $\bigoplus_{i\leqslant k}\mathcal{V}_i$, the subspace of $\mathcal{V}$ spanned by commutators of weight less than or equal to k. We can define the associated projection $p_k : \mathcal{V} \to \bigoplus_{i\leqslant k}\mathcal{V}_i$, and its composition with the restriction of γ_W to $\bigoplus_{i\leqslant k}\mathcal{V}_i$, denoted by $\gamma_k := p_k \circ \gamma_W|_{\bigoplus_{i\leqslant k}\mathcal{V}_i}$. Thanks to Proposition 4.2.31, γ_k is a representation with matrices given by the rows and columns corresponding to the blocks $B_{s,s}$ for $s \leqslant k$. Moreover $\gamma_k(\chi_{iI}) = Id$ for any comb-tree χ_{iI} with $\deg(\chi_{iI}) > k$. Hence we have $\gamma_k(\beta) = \gamma_k(\beta')$ for β' defined by:

$$\beta' = \prod_{\deg(\chi_{iJ})\leqslant k} \chi_{iJ}^{\nu_{iJ}}.$$

Now we show by strong induction on the degree k of χ_{iJ} that $\nu_{iJ} = 0$. For the base case $k = 1$, we take $i_0 \in \{1, \ldots, n\}$. Then using Lemma 4.2.33 iteratively and the fact that $\gamma_1(\chi_{iI}) = Id$ if $\deg(\chi_{iI}) > 1$, we obtain:

$$\gamma_1(\beta')(i_0) = \gamma_1\left(\prod_{1\leqslant i\neq j\leqslant n} \chi_{ij}^{\nu_{ij}}\right)(i_0) = (i_0) - \sum_{i_1<i_0} \nu_{i_0 i_1}\cdot(i_1 i_0) + \sum_{i_0<i_1} \nu_{i_0 i_1}\cdot(i_0 i_1).$$

Since $\beta \in \ker(\gamma)$, we have that $\gamma_1(\beta)(i_0) = (i_0)$, and this implies that $\nu_{i_0 i_1} = 0$ for any $i_1 \in \{1, \ldots, n\}$. To prove that $\nu_{iJ} = 0$ for any χ_{iJ} of degree k we use the strong induction hypothesis, we get then:

$$\beta' = \prod_{\deg(\chi_{iJ})\leqslant k} \chi_{iJ}^{\nu_{iJ}} = \prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu_{iJ}}.$$

Thus thanks to Lemma 4.2.33 again and the fact that $\gamma_k(\chi_{iI}) = Id$ if $\deg(\chi_{iI}) > k$, we finally obtain for all $i_0 \in \{1, \ldots, n\}$ that:

$$\gamma_k(\beta')(i_0) = \gamma_k\left(\prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu_{iJ}}\right)(i_0) = (i_0) - \sum_{\substack{J=(i_1,\ldots,i_k)\\ i_1<i_0}} \nu_{i_0 J}\cdot(J i_0) + \sum_{\substack{J=(i_1,\ldots,i_k)\\ i_0<i_1\\ i_1=\min(J)}} \nu_{i_0 J}(i_0 J) + S,$$

where S is a sum of commutators of the form $(i_0, i_1, \cdots, i_k)$ with $i_1 \neq \min\{i_1, \ldots, i_k\}$. So in particular no commutator in S occurs in the two above sums. Now, since $\beta \in \ker(\gamma)$ we have that $\gamma_k(\beta)(i_0) = (i_0)$.

[3] Roughly speaking, this term arises by 'tacking the term T_H' in each occurrence of the IHX relation, see Figure 4.9.

Thus considering the terms of the first sum we have that $\nu_{i_0 J} = 0$ for any $J = (i_1, \ldots, i_k)$ such that $i_1 < i_0$. Moreover, considering the second sum, we have that $\nu_{i_0 J} = 0$ for any $J = (i_1, \ldots, i_k)$ such that $i_0 < i_1$. Finally $\nu_{iJ} = 0$ for any χ_{iJ} of degree k which concludes the induction and completes the proof. □

Corollary 4.2.35. *The normal form is unique in hWB_n, i.e., if $\beta = \prod \chi_{iJ}^{\nu_{iJ}} = \prod \chi_{iJ}^{\nu'_{iJ}}$ are two normal forms of β for a given order on the set of positive comb-trees, then $\nu_{iJ} = \nu'_{iJ}$ for any integer i and any sequence J.*

Proof. Recall that for a given integer k, the sub-representation γ_k from the previous proof satisfies $\gamma_k(\beta) = \gamma_k(\beta')$ for β' defined by:

$$\beta' = \prod_{\deg(\chi_{iJ}) \leqslant k} \chi_{iJ}^{\nu_{iJ}} = \prod_{\deg(\chi_{iJ}) \leqslant k} \chi_{iJ}^{\nu'_{iJ}}.$$

As in the proof of Theorem 4.2.34, we proceed by strong induction on the degree that $\nu_{iJ} = \nu'_{iJ}$, the base case being strictly similar. For the inductive step, note that by Proposition 4.1.11, a comb-tree of degree k commutes with any comb-tree up to higher degree w-trees. Hence if χ_{iJ} is a comb-tree of degree k then $\gamma_k(\chi_{iJ})$ commutes with $\gamma_k(\chi_I)$ for any comb-trees χ_I. In particular we get:

$$\begin{aligned}\gamma_k(\beta')(i_0) =& \gamma_k\left(\prod_{\deg(\chi_{iJ})<k} \chi_{iJ}^{\nu_{iJ}}\right) \circ \gamma_k\left(\prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu_{iJ}}\right)(i_0),\\ =& \gamma_k\left(\prod_{\deg(\chi_{iJ})<k} \chi_{iJ}^{\nu'_{iJ}}\right) \circ \gamma_k\left(\prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu'_{iJ}}\right)(i_0).\end{aligned}$$

By induction hypothesis $\nu_{iJ} = \nu'_{iJ}$ for all χ_{iJ} such that $\deg(\chi_{iJ}) < k$. Hence, multiplying by the inverse of $\gamma_k\left(\prod_{\deg(\chi_{iJ})<k} \chi_{iJ}^{\nu_{iJ}}\right)$ we obtain the equality:

$$\gamma_k\left(\prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu_{iJ}}\right)(i_0) = \gamma_k\left(\prod_{\deg(\chi_{iJ})=k} \chi_{iJ}^{\nu'_{iJ}}\right)(i_0).$$

Finally, with Lemma 4.2.33 we obtain:

$$(i_0) - \sum_{\substack{J=i_1,\ldots,i_k\\ i_1<i_0}} \nu_{i_0 J}\cdot(Ji_0) + \sum_{\substack{J=i_1,\ldots,i_k\\ i_0<i_1\\ i_1=\min(J)}} \nu_{i_0 J}(i_0 J) + S = (i_0) - \sum_{\substack{J=i_1,\ldots,i_k\\ i_1<i_0}} \nu'_{i_0 J}\cdot(Ji_0) + \sum_{\substack{J=i_1,\ldots,i_k\\ i_0<i_1\\ i_1=\min(J)}} \nu'_{i_0 J}(i_0 J) + S',$$

where S and S' are sums of commutators of the form $(i_0,i_1,\cdots,i_k)$ with $i_1 \neq \min\{i_1,\ldots,i_k\}$. In particular, they are distinct from the commutators in the other sums. Therefore, we deduce from the above equality that $\nu_{iJ} = \nu'_{iJ}$ for all χ_{iJ} such that $\deg(\chi_{iJ}) = k$, which concludes the induction as well as the proof. □

4.3 The torsion problem revisited

In this section, the torsion problem in the homotopy braid group hB_n is addressed again. But the welded context (in which the classical braids are embedded) provides a better understanding of the torsion. Following the reasoning in Section 2.4, along with welded techniques, we will eventually show the absence of torsion in hB_n for all n (Theorem 4.3.8).

As a first result we can already state the welded analogue of Theorem 2.4.16.

Theorem 4.3.1. *The pure homotopy welded braid group hWP_n is torsion-free for any $n \in \mathbf{N}$.*

Proof. The proof follows from the global shape predicted by Proposition 4.2.31 of the matrix corresponding to the image $\gamma_W(\theta)$ of any $\theta \in hWP_n$ by the representation γ_W. It is a lower triangular matrix which contains a diagonal of 1's, and therefore satisfies $\gamma_W(\theta)^m = \mathrm{Id}$ for some integer m if and only if $\gamma_W(\theta) = \mathrm{Id}$. Finally, by Theorem 4.2.34, the injectivity of γ_W implies that if $\theta^m = \mathbf{1}$ for some pure homotopy welded braid θ and some integer m then $\theta = \mathbf{1}$. □

Remark 4.3.2. *It is well known to the experts that hWP_n is torsion-free. This can indeed also be shown using the additivity of Milnor numbers.*

Let us set $\lambda_n \in hWB_n$ the homotopy welded braid, illustrated in Figure 4.23, given by

$$\lambda_n = \rho_1 \rho_2 \cdots \rho_{n-1}.$$

We denote by τ_n the cycle $(n\ n-1\ \cdots\ 2\ 1) = \pi(\lambda_n)$ associated to λ_n. When the value of n is clear from the context, it will be omitted in the notation.

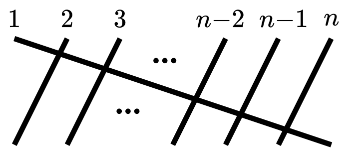


Figure 4.23: The homotopy welded braid λ_n.

Lemma 4.3.3. *Let $i \in \{1, \ldots, n\}$ and let I be a sequence of non-repeated indices in $\{1, \ldots, n\}\backslash\{i\}$. Suppose further that χ_{iI} is a comb-tree of degree d. Then, the conjugate $\lambda\chi_{iI}\lambda^{-1}$ is link-homotopic to a product of degree d comb-trees, all having their head on the component $\tau^{-1}(i)$.*

Proof. We first use an arrow isotopy to slide χ_{iI} through λ and then simplify λ with λ^{-1} with a welded isotopy. This turns χ_{iI} into a new w-tree of degree d with head on component $\tau^{-1}(i)$. Then using Lemma 4.2.5 we turn it, up to homotopy, into a product of degree k comb-trees all having their head on the component $\tau^{-1}(i)$. □

Lemma 4.3.4. *Let $\beta \in hWB_n$ be a homotopy welded braid, whose associated permutation is an n-cycle. Then β is conjugate to the product $\theta\lambda$ with a pure homotopy welded braid $\theta \in hWP_n$ whose normal form*

$$\theta = \prod \chi_I^{\nu_I}$$

contains only comb-trees with head on the n-th component.

Proof. Up to conjugation, we can suppose that $\pi(\beta) = \tau$. Then $\beta = \theta\lambda$ with $\theta = \beta\lambda^{-1}$ a pure homotopy welded braid with normal form given by:

$$\theta = \chi_{I_1}^{\nu_1}\chi_{I_2}^{\nu_2}\cdots\chi_{I_m}^{\nu_m}. \tag{4.1}$$

Let us assume that $\nu_i \neq 0$ for some χ_{I_i} whose head is not on the n-th component. Let us further suppose that χ_{I_i} is of minimal degree, i.e., the head of all comb-trees of degree smaller than $\deg(\chi_{I_i})$ are on the n-th component. By Lemma 4.3.3 there exists some integer $l > 0$, such that the conjugate $\lambda^l(\chi_{I_i})\lambda^{-l}$ is link-homotopic to a product of comb-trees with head on the n-th component and with same degree as χ_{I_i}. We consider β', the conjugate of β given by:

$$\begin{aligned}
\beta' &= \left(\prod_{0\leqslant k<l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right)^{-1} \beta \left(\prod_{0\leqslant k<l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right),\\
&= \left(\prod_{0\leqslant k<l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right)^{-1} \theta \left(\prod_{0<k\leqslant l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right)\lambda,\\
&= \left(\prod_{0<k<l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right)^{-1} \chi_{I_i}^{-\nu_i}\theta \left(\prod_{0<k<l} \lambda^k\chi_{I_i}^{\nu_i}\lambda^{-k}\right)\left(\lambda^l\chi_{I_i}^{\nu_i}\lambda^{-l}\right)\lambda.
\end{aligned}$$

Now note that, according to Lemma 4.3.3, the conjugates $\lambda^k(\chi_{I_i})\lambda^{-k}$ for $0 < k < l$ can be seen as products of comb-trees with same degree as χ_{I_i}. Moreover, thanks to Lemma 4.1.11 two comb-trees commute up to higher degree w-trees, and by Lemma 4.2.5 we can assume that these higher degree w-trees are also comb-trees. Then in the previous expression, up to comb-trees of degree greater than that of χ_{I_i}, we can simplify the terms $\lambda^k(\chi_{I_i})\lambda^{-k}$ for $0 < k < l$ with their inverse to obtain:

$$\beta' = \chi_{I_i}^{-\nu_i}\theta\left(\lambda^l\chi_{I_i}^{\nu_i}\lambda^{-l}\right)\left(\prod_{\deg(\chi_{I_i})<\deg(\chi_I)} \chi_I\right)\lambda.$$

Since the factor $\chi_{I_i}^{\nu_i}$ appears in the normal form $\theta = \chi_{I_1}^{\nu_1}\chi_{I_2}^{\nu_2}\cdots\chi_{I_m}^{\nu_m}$, we can, using the same argument, express β' as follows:

$$\beta' = \left(\chi_{I_1}^{\nu_1}\cdots\chi_{I_{i-1}}^{\nu_{i-1}}\chi_{I_{i+1}}^{\nu_{i+1}}\cdots\chi_{I_m}^{\nu_m}\right)\left(\lambda^l\chi_{I_i}^{\nu_i}\lambda^{-l}\right)\left(\prod_{\deg(\chi_{I_i})<\deg(\chi_I)} \chi_I\right)\lambda.$$

Finally we denote by θ' the pure part of the product $\beta' = \theta'\lambda$, the last step consists in computing the normal form,

$$\theta' = \chi_{I_1}^{\nu_1'}\chi_{I_2}^{\nu_2'}\cdots\chi_{I_m}^{\nu_m'}. \tag{4.2}$$

Starting with

$$\theta = \left(\chi_{I_1}^{\nu_1}\cdots\chi_{I_{i-1}}^{\nu_{i-1}}\chi_{I_{i+1}}^{\nu_{i+1}}\cdots\chi_{I_m}^{\nu_m}\right)\left(\lambda^l\chi_{I_i}^{\nu_i}\lambda^{-l}\right)\left(\prod_{\deg(\chi_{I_i})<\deg(\chi_I)} \chi_I\right),$$

this is achieved by rearranging comb-trees degree by degree as it is done in the proof of Theorem 4.2.7. Let us compare the exponents ν'_j and ν_j associated to the two normal forms (4.1) and (4.2). First, if $\deg(\chi_{I_j}) < \deg(\chi_{I_i})$ then $\nu'_j = \nu_j$ since no new comb-tree of degree lower than χ_{I_i} appeared in the procedure. Second, it is clear that the exponent ν'_i associated to χ_{I_i} in (4.2) is now trivial, i.e., $\nu'_i = 0$. Finally, $\nu'_j = \nu_j$ for almost all other comb-trees χ_{I_j} of degree equal to $\deg(\chi_{I_i})$. The only exceptions come from the conjugate $\lambda^l \chi_{I_i}^{\nu_i} \lambda^{-l}$ and concern comb-trees whose head is on the n-th component.

In summary, the exponents of χ_{I_j} of degree $\deg(\chi_{I_j}) \leqslant \deg(\chi_{I_i})$ whose head is not in the $n-$th component remain the same, except for the exponent of χ_{I_i} which has become zero. Hence, by repeating the above argument, we eventually obtain another conjugate of β of the form $\tilde{\theta}\lambda$ such that, any comb-tree of degree lower than or equal to $\deg(\chi_{I_i})$ in the normal form of $\tilde{\theta}$ has its head on the n-th component. Moreover, since all w-trees of degree greater than n are trivial up to homotopy, by proceeding by increasing degree, we can get rid of all comb-trees whose head is not on the n-th component and finally obtain the desired conjugate. □

As mentioned in Proposition 4.2.23 the group hB_n appears as the subgroup of hWB_n generated by the Artin generators σ_i for $1 \leqslant i < n$. We say that a homotopy welded braid is a **classical braid** if it belongs to this subgroup. In the following lemma we give a new characterization of the torsion in hB_n using this notion of classical braid.

Lemma 4.3.5. *There is torsion in hB_n if and only if for some prime number $p \leqslant n$ the braid $\lambda_p \in hWB_p$ given by $\lambda_p = \rho_1\rho_2 \cdots \rho_{p-1}$ is conjugate to a classical braid.*

Proof. According to Lemma 2.4.17 if there is torsion in hB_n, we can find a torsion element β of order p in hB_p, for some prime number p, which we regard as a classical braid β in hWB_p. Moreover Theorem 4.3.1 implies that $\pi(\beta) \neq \mathrm{Id}$ but we know that $\pi(\beta)^p = \mathrm{Id}$. In other words, $\pi(\beta)$ is a torsion element of order p in the p-th symmetric group meaning that it is a p-cycle. Then by Lemma 4.3.4, β is conjugate to the product $\theta\lambda$ where the normal form

$$\theta = \prod \chi_{pI}^{\nu_{pI}},$$

only contains comb-trees with head on the p-th component (for clarity, here and throughout the remainder of the proof, we denote λ_p simply by λ). Moreover by Lemma 4.3.3, for any integer $k \in \{1, \dots p-1\}$, the conjugates $\lambda^k\theta\lambda^{-k}$ are products of comb-trees, none of which have their head on the p-th component. Hence by Lemma 4.2.33, we have that $\gamma_W\left(\lambda^k\theta\lambda^{-k}\right)(p) = (p)$ if $0 < k < p$. In particular:

$$\begin{aligned}\gamma_W\left((\theta\lambda)^p\right)(p) &= \gamma_W\left(\theta\left(\lambda\theta\lambda^{-1}\right)\left(\lambda^2\theta\lambda^{-2}\right)\cdots\left(\lambda^{p-1}\theta\lambda^{1-p}\right)\lambda^p\right)(p),\\ &= \gamma_W(\theta)(p).\end{aligned}$$

On the other hand, since β is a torsion element, $\beta^p = (\theta\lambda)^p = \mathbf{1}$, which implies that

$$\gamma_W\left((\theta\lambda)^p\right)(p) = \gamma_W(\mathbf{1})(p) = (p).$$

By combining the two previous equality we deduce that $\gamma_W(\theta)(p) = (p)$. Moreover by Lemma 4.2.33 again, we also have that $\gamma_W(\theta)(k) = (k)$ for any $k < p$. In particular, using Remark 4.2.32, we see that $\gamma_W(\theta)$ is the identity, and by injectivity of γ_W, the braid θ is as well. Consequently, the classical braid β is conjugate to λ, and thus, the first half of the proof is complete.

To show the converse, we use the fact that any conjugate of λ is a torsion element of order p in hWB_p and that, consequently, the braid given by the same expression in hB_n is also a torsion element. □

The end of this section consists in showing that for any integer n, the braid λ has no classical braid as conjugate. To do so, we will first recall the usual characterization from [HL90] of classical braids in terms of automorphisms of the reduced free group. In fact, we will take a slightly different view by using the *reduced Magnus expansion* of the proof of Theorem 2.3.12. Recall that this is the homomorphism $\tilde{M}$ from the reduced free group into the polynomial algebra in non-commuting variables $X_1, \ldots, X_n$ in which monomials $X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_k}$ vanish if $\alpha_i = \alpha_j$ for some $i \neq j$. The image of a generator x_i is given by the polynomial $\tilde{M}(x_i) = 1 + X_i$. In [Yur08, Theorem 7.11] and [Dar23, Corollary 1.13], it is proved that $\tilde{M}$ is injective, so it is an isomorphism onto its image, which we denote by $\mathcal{I}_n$. Note that $\mathcal{I}_n$ is the group generated by $1 + X_i$ for $i \in \{1, \ldots, n\}$. We can then define a representation of the homotopy welded braid group $Z : hWB_n \to Aut(\mathcal{I}_n)$ given by:

$$Z : \beta \mapsto \tilde{M} \circ \zeta_h(\beta) \circ \tilde{M}^{-1},$$

where ζ_h is the homotopy welded Artin representation defined in Definition 4.2.18. For later use, let us compute the image of the Artin generators σ_i by the representation Z,

$$Z(\sigma_i) : \begin{cases} 1 + X_i & \mapsto & 1 + X_{i+1}, \\ 1 + X_{i+1} & \mapsto & 1 + X_i + X_iX_{i+1} - X_{i+1}X_i, \\ 1 + X_k & \mapsto & 1 + X_k, & \text{if } k \notin \{i, i+1\}, \end{cases}$$

and the image of the virtual generator ρ_i,

$$Z(\rho_i) : \begin{cases} 1 + X_i & \mapsto & 1 + X_{i+1}, \\ 1 + X_{i+1} & \mapsto & 1 + X_i, \\ 1 + X_k & \mapsto & 1 + X_k, & \text{if } k \notin \{i, i+1\}. \end{cases}$$

We also compute the image of the braid $\lambda = \rho_1 \cdots \rho_{n-1}$ by the representation Z:

$$Z(\lambda) : \begin{cases} 1 + X_i & \mapsto & 1 + X_{i+1}, & \text{if } i < n, \\ 1 + X_n & \mapsto & 1 + X_1, \end{cases}$$

More simply, we can think of $Z(\lambda)$ as the automorphism permuting the variables X_i in the full ring. Let us now state a property, inspired from [HL90, Theorem 1.7], on classical braids in terms of automorphisms of $\mathcal{I}_n$.

Lemma 4.3.6. *Let $\beta \in hWB_n$ be a homotopy welded braid. If β is a classical braid then*

$$Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_n)\big) = \tilde{M}(x_1x_2\cdots x_n).$$

Proof. Let us first recall the expression of the homotopy welded Artin representation on the classical Artin generators σ_i:

$$\zeta_h(\sigma_i) : \begin{cases} x_i & \mapsto & x_{i+1}, \\ x_{i+1} & \mapsto & x_{i+1}^{-1}x_ix_{i+1}, \\ x_k & \mapsto & x_k & \text{if } k \notin \{i, i+1\}, \end{cases}$$

It is clear by computation that

$$\zeta_h(\sigma_i)(x_1x_2\cdots x_n) = x_1x_2\cdots x_n$$

for any classical Artin generator σ_i. In particular, this implies the well-known fact that if $\beta \in hWB_n$ is a classical braid, then

$$\zeta_h(\beta)(x_1x_2\cdots x_n) = x_1x_2\cdots x_n,$$

or equivalently,

$$Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_n)\big) = \tilde{M}(x_1x_2\cdots x_n).$$

□

Let us make the observation that $\tilde{M}(x_1x_2\cdots x_n)$ contains exactly one monomial of degree n, given by $X_1X_2\cdots X_n$. Furthermore $Z(\lambda)(X_1X_2\cdots X_n) \neq X_1X_2\cdots X_n$, so $Z(\lambda)\big(\tilde{M}(x_1x_2\cdots x_n)\big) \neq \tilde{M}(x_1x_2\cdots x_n)$. In the following lemma, we go a little further in describing elements that are not fixed points of $Z(\lambda)$.

Lemma 4.3.7. *The automorphism $Z(\lambda)$ has no fixed point of the form $Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_n)\big)$ for any $\beta \in hWB_n$.*

Proof. Let us denote by $\mathcal{A}$ the polynomial algebra in non-commuting variables $X_1, \ldots, X_n$ in which monomials $X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_k}$ vanish if $\alpha_i = \alpha_j$ for some $i \neq j$. Now, consider the additive homomorphism $F : \mathcal{A} \to \mathbf{Z}$ defined on the monomials. by:

$$F(X_{\alpha_1}X_{\alpha_2}\cdots X_{\alpha_k}) = \begin{cases} 0 & \text{if } k < n, \\ 1 & \text{if } k = n. \end{cases}$$

In other words, the homomorphism F sends a polynomial to the sum of the coefficients of its monomials of degree n. Let us note on the one hand that $F\big(Z(\rho_i)(W)\big) = F(W)$ and $F\big(Z(\sigma_i)(W)\big) = F(W)$ for any $W \in \mathcal{I}_n$ and any i. This is clear for ρ_i, which simply permutes the variables X_i and X_{i+1}. It is less clear for σ_i, which, after the permutation, substitutes X_i with $X_i + X_iX_{i+1} + X_{i+1}X_i$, potentially introducing new monomials of degree n. However, these monomials appear in pairs and with opposite signs and thus do not change the value of F. So it is clear that

$$F\big(Z(\beta)(W)\big) = F(W),$$

for any $\beta \in hWB_n$ and any $W \in \mathcal{I}_n$. Moreover note that,

$$\begin{aligned} F\big(\tilde{M}(x_1x_2\cdots x_n)\big) &= F\big((1+X_1)(1+X_2)\cdots(1+X_n)\big), \\ &= 1, \end{aligned}$$

hence $F\Big(Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_n)\big)\Big) = 1$ for any $\beta \in hWB_n$. But on the other hand, $Z(\lambda)$ acts on the nontrivial monomials by permuting the variables X_i cyclically, and the orbits of the action are of cardinality n. Therefore, if an element $W \in \mathcal{I}_n\backslash\{1\}$ satisfies that $Z(\lambda)(W) = W$ then it has to verify

$$F(W) \equiv 0 \mod [n].$$

Hence such a fixed point W of $Z(\lambda)$ cannot be of the form $Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_n)\big)$ for any $\beta \in hWB_n$ □

We can now state the final theorem of this section.

Theorem 4.3.8. *The homotopy braid group hB_n is torsion-free for any number of components n.*

Proof. Suppose by contradiction that there is a torsion element in hB_n. By Lemma 4.3.5 there exist a prime number $p \leqslant n$ and some braid $\beta \in hWB_p$ such that $\beta^{-1}\lambda_p\beta$ is a classical braid, where $\lambda_p = \rho_1\rho_2\cdots\rho_{p-1} \in hWB_p$. In other words, according to Lemma 4.3.6 this conjugate must satisfy,

$$Z\big(\beta^{-1}\lambda\beta\big)\big(\tilde{M}(x_1x_2\cdots x_p)\big) = \tilde{M}(x_1x_2\cdots x_p),$$

or equivalently,

$$Z(\lambda) \circ Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_p)\big) = Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_p)\big).$$

This implies that $Z(\beta)\big(\tilde{M}(x_1x_2\cdots x_p)\big)$ is a fixed point of $Z(\lambda)$, which yields a contradiction by Lemma 4.3.7. ☐

It follows from Theorem 4.3.8 the well known result that the standard braid group B_n is torsion-free for all n. To prove this corollary, we need the following well-known lemma, which essentially goes back to Artin:

Lemma 4.3.9. *The pure braid group P_n is torsion-free for any number of components n.*

Proof. The pure braid group P_n can be expressed as a semi-direct product of free groups, known as the *Artin normal form*. The procedure to obtain this normal form is known as *braid combing* and is presented in [Art47]. Therefore, since free groups are torsion-free, it simply follows that P_n is torsion-free. ☐

We recover in this way a result of E. Fadell and L. Neuwirth (see Remark 4.3.11).

Corollary 4.3.10. *The braid group B_n is torsion-free for any number of components n.*

Proof. Let us consider the projection $p : B_n \to hB_n$. Since hB_n is torsion-free (Theorem 4.3.8), any torsion element in B_n must belong to the kernel $K := \ker(p)$. However, it is clear from Proposition 2.2.2 that $K \subset P_n$, thus K is torsion-free by Lemma 4.3.9 and the proof is complete. ☐

Remark 4.3.11. *The study of torsion in braid groups dates back to E. Fadell and L. Neuwirth in 1962. Building upon topological methods, they show in [FN62, Theorem 8] that B_n is torsion-free for every n. Subsequently, P. Dehornoy establishes the stronger property that B_n is left-orderable, in [Deh94] which in particular implies that it is torsion-free. It should be noted that the question of orderability for the homotopy braid group hB_n remains open, constituting a future research direction that we intend to explore.*

Chapter 5

Homology cobordisms and homology cylinders

This chapter focuses on the study of *homology cobordisms*. Exploratory work is conducted to define a notion of link-homotopy within this context. This pursuit holds significance as string-links and braids share many common features with homology cobordisms, as discussed at the beginning of Section 5.2. The chapter begins by defining the framework of homology cobordisms in Section 5.1 and subsequently explores several tentative definitions for link-homotopy in Section 5.2 and 5.3.

5.1 General definition

Let us denote by Σ a compact connected oriented surface of genus g. We assume for simplicity that Σ has exactly one boundary component. Let us recall the definition of the *mapping class group*.

Definition 5.1.1. *The* **mapping class group** *of the surface Σ, denoted by $\mathcal{M}(\Sigma)$, is the group of isotopy classes of self-homeomorphisms of Σ that leave the boundary pointwise invariant.*

Definition 5.1.2. *Let c be a simple closed curve on Σ not necessarily oriented. We choose a closed regular neighborhood N of c in Σ and we identify it with $S^1 \times [0,1]$ in such a way that orientations are preserved. Then, the* **Dehn twist** *along c is the homeomorphism $T_c : \Sigma \to \Sigma$ defined by:*

$$T_c(x) = \begin{cases} x & \text{if } x \notin N, \\ \left(e^{2i\pi(\theta+r)}, r\right) & \text{if } x = (e^{2i\pi(\theta)}, r) \in N. \end{cases}$$

We illustrate the effect of a Dehn twist on a small segment in Figure 5.1.

Dehn twists generate the mapping class group as stated in the following theorem.

Theorem 5.1.3. *[Deh38] The mapping class group $\mathcal{M}(\Sigma)$ is generated by Dehn twists along curves which are non-separating (i.e., the surface given by Σ with the curve removed, has a single connected component), or parallel to a boundary component.*

The mapping class group of the surface Σ acts in a canonical way on the fundamental group $\pi := \pi_1(\Sigma, *)$ based at a point $*$ on the boundary of Σ. The induced homomorphism

$$\rho : \mathcal{M}(\Sigma) \to \mathrm{Aut}(\pi),$$

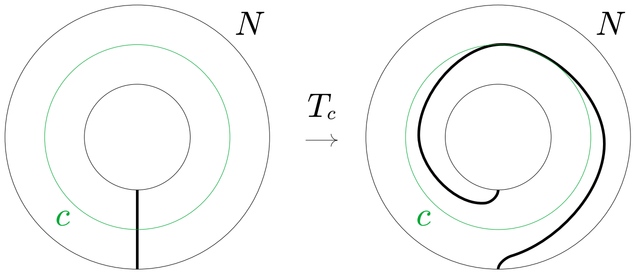


Figure 5.1: The Dehn twist T_c along the simple closed curve c.

studied by Dehn–Nielsen is known to be injective. Moreover, for each $k \geqslant 0$, it induces a representation

$$\rho_k : \mathcal{M}(\Sigma) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$$

where $\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \Gamma_3\pi \supset \cdots$, denotes the lower central series of π; i.e., the sequence of subgroups defined by :

$$\begin{cases} \Gamma_1\pi = \pi, \\ \Gamma_{k+1}\pi = [\Gamma_k\pi,\pi]. \end{cases}$$

We refer to those representations as the **nilpotent Dehn–Nielsen representations**. The **Johnson filtration** of the mapping class group is the decreasing sequence of subgroups

$$\mathcal{M}(\Sigma) = \mathcal{M}(\Sigma)[0] \supset \mathcal{M}(\Sigma)[1] \supset \mathcal{M}(\Sigma)[2] \supset \mathcal{M}(\Sigma)[3] \supset \cdots , \tag{5.1}$$

where $\mathcal{M}(\Sigma)[k]$ denotes the kernel of ρ_k for all $k \geqslant 1$.

Definition 5.1.4. *The first subgroup in this filtration, denoted by $\mathcal{M}(\Sigma)[1]$, is referred to as the **Torelli group** of the surface Σ. In simple terms, it is the subgroup of homeomorphisms of Σ that act trivially on its homology.*

Theorem 5.1.5. *[Bir71, Pow78] The Torelli group of Σ is generated by two types of Dehn twists:*

- separating twists*: Dehn twists along separating curves, i.e., curves that divide the surface into two sub-surfaces.*

- bounding pair maps*: The composition of a Dehn twist along a non-separating curve and the inverse Dehn twist along another non-separating curve, disjoint from the first one but having the same homology class.*

Let us now define the main objects of the section : *homology cylinders.*

Definition 5.1.6. *A* **homology cobordism** *over Σ is a pair (C, i), where C is a compact connected oriented 3-manifold and $i : \partial(\Sigma \times [-1,1]) \to \partial C$ is an orientation-preserving homeomorphism such that the inclusion $i_\pm : \Sigma \to M$ defined by $x \mapsto i(x, \pm 1)$ induce isomorphisms $H_*(\Sigma; \mathbf{Z}) \to H_*(C; \mathbf{Z})$. Thus the 3-manifold C is a homology cobordism between $\partial_+C := i_+(\Sigma)$ and $\partial_-C := i_-(\Sigma)$.*

For simplicity, we often denote the homology cobordism (C,i) by C, in particular, we denote the trivial homology cobordism $(\Sigma \times [-1,1], Id)$ simply by $\Sigma \times [-1,1]$. We call **homology cylinders** of Σ the homology cobordisms for which the composition $(i_-)^{-1} \circ (i_+)$ is the identity of $H_*(\Sigma; \mathbf{Z})$.

We say that two homology cobordisms are homeomorphic if there is an orientation-preserving homeomorphism $f : C \to C'$ such that $f|_{\partial C} \circ i = i'$. The composition '$\circ$' of two homology cobordisms C and C' is defined by 'stacking' C' on top of C, i.e., we define

$$C \circ C' := C \cup_{i_+ \circ (i'_-)^{-1}} C',$$

with $\partial_-(C \circ C') = \partial_-(C)$ parameterized by i_- and $\partial_+(C \circ C') = \partial_+(C')$ by i'_+. With this operation, the set of homeomorphism classes of homology cobordisms of Σ, denoted by $\mathcal{C}(\Sigma)$, forms a monoid. The set of homeomorphism classes of homology cylinders of Σ, denoted by $\mathcal{IC}(\Sigma)$ is a submonoid of $\mathcal{C}(\Sigma)$. Moreover, the mapping class group is embedded in $\mathcal{C}(\Sigma)$ using the **mapping cylinder construction** $\iota : \mathcal{M}(\Sigma) \to \mathcal{C}(\Sigma)$, defined as follows:

$$\iota(\phi) := \Big(\Sigma \times [-1,1], \big(\mathrm{Id} \times \{-1\}\big) \cup (\partial\Sigma \times Id) \cup \big(\phi \times \{1\}\big)\Big).$$

In fact, the image of ι is the group of invertibles in the monoid $\mathcal{C}(\Sigma)$, see [HM12, Proposition 2.4]. Similarly, the Torelli group embeds in $\mathcal{IC}(\Sigma)$ through the same mapping cylinder construction.

Thanks to Stallings' theorem [Sta65] we can extend the nilpotent Dehn–Nielsen representations to homology cobordisms.

Theorem 5.1.7. *[GL05, Theorem 3] For any homology cobordism $C \in \mathcal{C}(\Sigma)$ and any $k \in \mathbf{N}$, the composition $(i_-)^{-1} \circ (i_+)$ induces a homomorphism:*

$$\rho_k : \mathcal{C}(\Sigma) \to Aut(\pi/\Gamma_{k+1}\pi).$$

Note that the restriction of this morphism to $\iota(\mathcal{M}(\Sigma))$ coincides with the previously defined Dehn–Nielsen representation, justifying the continued use of the same notation. Furthermore, if we denote by $C(\Sigma)[k] := \ker(\rho_k)$, we obtain a decreasing sequence of submonoids:

$$\mathcal{C}(\Sigma) = \mathcal{C}(\Sigma)[0] \supset \mathcal{C}(\Sigma)[1] \supset \mathcal{C}(\Sigma)[2] \supset \mathcal{C}(\Sigma)[3] \supset \cdots$$

which extends the filtration (5.1), and is still referred to as the *Johnson filtration*.

5.2 Link-homotopy in an algebraic way

There is a strong analogy between homology cobordisms and string-links. Firstly, the pure braid group (which can be defined as the mapping class group of the punctured disk), forms the subgroup of invertibles in the string-links monoid, as does the mapping class group of Σ for the monoid $\mathcal{C}(\Sigma)$. Moreover, as discussed in [GL05, Remark 5.3], $\mathcal{C}(\Sigma)$ can be converted into a group by considering their homology cobordism classes, similarly as string-links do up to concordance. Finally, Milnor string-links invariants, appears as the analogues of the homomorphisms ρ_k, as pointed in [GL05, Remark 2.6] and [Hab00a]. See [Ver21, Section 2.4] for a good exposition of this so-called *Milnor-Johnson correspondence*. In light of this correspondence, it is thus natural to investigate an analogue of the link-homotopy relation of string-links in the context of homology cobordisms.

5.2.1 Reduced group

To establish a notion of link-homotopy for homology cylinders, or more generally for homology cobordisms, it is natural, at the algebraic level, to consider them as automorphisms of the reduced free group. Since the fundamental group of Σ, is a free group, Theorem 5.1.7 seems to be a good first step in this direction. Let us fix the family of generators x_1, y_1, …, x_g, y_g, illustrated in Figure 5.2.

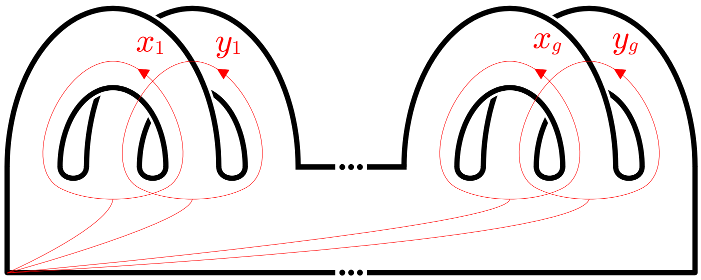


Figure 5.2: Generators x_1, y_1, …, x_g, y_g of π.

We recall from Proposition 1.2.5 the subgroup $J := J_\pi$ of π generated by commutators in x_1, y_1, …, x_n, y_n with repeats and the reduced quotient of π given by $\mathcal{R}\pi := \pi/J$. We make the observation that, for $k \geqslant 2g$, $\mathcal{R}\pi$ is given by the quotient of $\pi/\Gamma_{k+1}\pi$ by $J/\Gamma_{k+1}\pi$. This follows from the fact that any commutator of weight greater than $2g$ has repeats and therefore $\Gamma_{k+1}\pi \subset J$. Then for $k \geqslant 2g$ we may hope that ρ_k will be turned into a homomorphism from $\mathcal{C}(\Sigma)$ to $Aut(\mathcal{R}\pi)$. The only condition that must be verified to do so is that for any homology cobordism C we have:

$$\rho_k(C)(J/\Gamma_{k+1}\pi) = J/\Gamma_{k+1}\pi. \tag{5.2}$$

Moreover, if the homology cobordism C_ϕ is given by a mapping cylinder construction, i.e., $C_\phi = \iota(\phi)$ for some $\phi \in \mathcal{M}(\Sigma)$, then it induces an automorphism of the free group $\rho(C_\phi) := \rho(\phi)$, and to see it as an element of $Aut(\mathcal{R}\pi)$, we simply need:

$$\rho(C_\phi)(J) = J. \tag{5.3}$$

But this is not the case in general as the following two counter-examples show. Let us set $\Sigma_{2,1}$ the surface with genus $g = 2$ and one boundary component.

Counter-example 5.2.1. *We consider the mapping cylinder $\iota(T_c)$ of the Dehn twist T_c along the simple closed curve c, illustrated in Figure 5.3.*

This element, denoted by C_c, seen as an automorphism of the fundamental group, is given by:

$$\begin{array}{cccc} \rho(C_c) & : & \pi & \to & \pi \\ & & x_1 & \mapsto & x_1, \\ & & y_1 & \mapsto & y_1, \\ & & x_2 & \mapsto & x_2, \\ & & y_2 & \mapsto & y_2x_2. \end{array}$$

We compute the image of the commutator with repeats $[[x_1,y_2],y_2] \in J$ by the automorphism $\rho(C)$:

$$\rho(C_c)([[x_1,y_2],y_2]) = [[x_1,y_2x_2],y_2x_2].$$

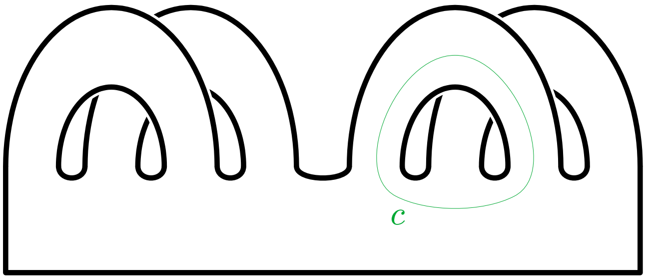


Figure 5.3: The closed curve c.

This element, seen as an element of the reduced free group, has normal form (Definition 1.2.7) given by:

$$\rho(C_c)([[x_1,y_2],y_2]) \equiv [[x_1,x_2],y_2][[x_1,y_2],x_2],$$

By uniqueness of the normal form (Theorem 1.2.10), the image $\rho(C_c)([[x_1,y_2],y_2])$ *does not belong to* J*. Therefore condition* (5.3) *is not verified, and we cannot see* C_c *as an automorphism of the reduced free group.*

But the homology cobordism in Counter-example 5.2.1 is not a homology cylinder, and one might expect the desired construction to be satisfied by these objects. However, as shown in Counter-example 5.2.2, we still have the same problem for homology cylinders.

Counter-example 5.2.2. *Consider the homology cylinder* $C_{a,b} := \iota(T_a^{-1} \circ T_b)$*, where* T_a *and* T_b *are the Dehn twists along the simple closed curves* a *and* b*, illustrated in Figure 5.4. Note that* (a,b) *forms a genus one bounding pair.*

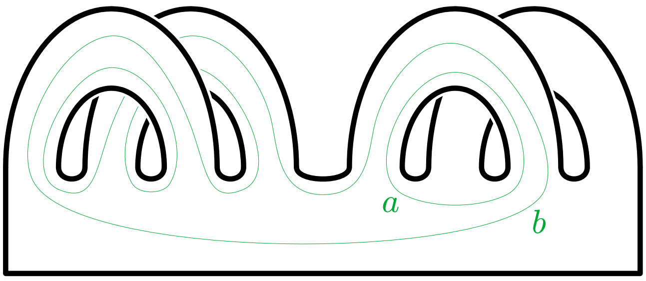


Figure 5.4: The bounding pair (a,b).

This homology cobordism, induces an automorphism on the fundamental group given by:

$$\begin{array}{rcccc} \rho(C_{a,b}) & : & \pi & \longrightarrow & \pi \\ & & x_1 & \longmapsto & x_2[y_1,x_1^{-1}]x_1[x_1^{-1},y_1]x_2^{-1} \\ & & y_1 & \longmapsto & x_2[y_1,x_1^{-1}]y_1[x_1^{-1},y_1]x_2^{-1} \\ & & x_2 & \longmapsto & x_2[y_1,x_1^{-1}]x_2[x_1^{-1},y_1]x_2^{-1} \\ & & y_2 & \longmapsto & y_2x_2[x_1^{-1},y_1]x_2^{-1} \end{array}$$

We have that:

$$\begin{aligned}\rho(C_{a,b})\Big(\big[[x_2,y_2],y_2\big]\Big) &\equiv \Big[\Big[\big[[x_1,y_1],x_2\big]x_2,y_2\big[x_2,[y_1,x_1]\big][y_1,x_1]\Big],y_2\big[x_2,[y_1,x_1]\big][y_1,x_1]\Big] && \text{mod } [J]\\ &\equiv \Big[\big[x_2,y_2[y_1,x_1]\big],y_2[y_1,x_1]\Big] && \text{mod } [J]\\ &\equiv \big[[x_2,y_2],[y_1,x_1]\big]\Big[\big[x_2,[y_1,x_1]\big],y_2\Big] && \text{mod } [J]\\ &\equiv \Big[\big[[x_1,y_1],y_2\big]x_2\Big]\Big[\big[[x_1,y_1],x_2\big]y_2\Big]^{-2} && \text{mod } [J]\end{aligned}$$

and once again we come across a normal form which is not that of the trivial element. Hence this homology cobordism does not induce an automorphism of the reduced fundamental group.

To address the issue highlighted by Counter-examples 5.2.1 and 5.2.2, we aim to identify another normal subgroup, denoted as $H \lhd \pi$, such that homology cobordisms can be viewed as automorphisms of the quotient group π/H. Our objective is twofold: firstly, we require that for any sufficiently large integer $k > 0$, the subgroup $\Gamma_k\pi$ is contained within H to leverage the applicability of Theorem 5.1.7. Secondly, to ensure the quotient's relevance in terms of link-homotopy, we seek a reduced-type quotient, meaning that some elements commute with their conjugates. As a result, in the subsequent section, we are led to consider the notion of *fully reduced group.*

5.2.2 Fully reduced group

In this section, we extend the definition of reduced groups in order to obtain a quotient which does not depend on a chosen family of generators.

Definition 5.2.3. *Let G be a group and let us define $H \lhd G$ to be the normal subgroup generated by elements of the form $[\omega,\lambda\omega\lambda^{-1}]$, for any $\omega,\lambda \in G$. We call* **fully reduced quotient***, the quotient G/H and we denote it by $\mathcal{R}_{\mathcal{F}}G$. Roughly speaking, $\mathcal{R}_{\mathcal{F}}G$ is the quotient of G in which any element commutes with its conjugates.*

Proposition 5.2.4. *For any group G and any $x,y,z \in \mathcal{R}_{\mathcal{F}}G$, the following equalities hold in $\mathcal{R}_{\mathcal{F}}G$:*

$$\begin{array}{ll}(1) & [x^{-1},y] = [x,y]^{-1} = [x,y^{-1}];\\ (2) & \big[[x,y],z\big]\big[[z,x],y\big]\big[[y,z],x\big] = 1;\\ (3) & \big[[x,y],z\big] = \big[[x,z],y\big]^{-1}.\end{array}$$

Proof. The first equality corresponds to the following fully reduced relations:

$$x^{-1}yxy^{-1} = yxy^{-1}x^{-1} = xy^{-1}x^{-1}y.$$

For the second relation we recall first the well known Hall–Witt identity:

$$x^{-1}\big[[y,x^{-1}],z\big]x \cdot z^{-1}\big[[x,z^{-1}],y\big]z \cdot y^{-1}\big[[z,y^{-1}],x\big]y = 1.$$

Then we turn all the factors into the desired ones. For example, applying (1) to $\big[[y,x^{-1}],z\big]$ yields $\big[[x,y],z\big]$, which is equal to its conjugate by x due to the fully reduced quotient relations. Finally, the last equality is derived from the observation that, on the one hand, the commutator $\big[[x,yz],yz\big]$ is trivial in $\mathcal{RF}G$ and, on the other hand, that we have the two equalities

$$\big[a,[b,c]\big] = [a,b]\big[c,[a,b]\big][a,c],$$
$$\big[[c,b],a\big] = \big[b,[c,a]\big][c,a][b,a],$$

for any a, b, c in any group G. Then, by iterating these relations we rewrite the commutator $\big[[x,yz],yz\big]$ as a product of commutators in x, y, and z. However, commutators of degree four or higher necessarily contain repetitions and are therefore trivial in $\mathcal{RF}G$. Consequently, we ultimately find that

$$\big[[x,yz],yz\big] = \big[[x,y],yz\big]\big[[x,y],z\big]\big[[x,z],y\big].$$

□

Proposition 5.2.5. *For any group G and any triple x, y, $z \in \mathcal{R}_{\mathcal{F}}G$ we have that:*

$$\big[[x,y],z\big]^3 = 1$$

Proof. We start with equality (3) from Proposition 5.2.4, and then apply (1) from Proposition 5.2.4 twice:

$$\begin{aligned}\big[[x,y],z\big] &= \big[[x,z],y\big]^{-1},\\ &= \big[[x,z]^{-1},y\big],\\ &= \big[[z,x],y\big].\end{aligned}$$

Similarly we have that

$$\big[[x,y],z\big] = \big[[y,z],x\big],$$

then we conclude using (2) from Proposition 5.2.4. □

In view of Proposition 5.2.5, it would seem that the fully reduced quotient is not suitable for our study. Too much information is lost, and it is unlikely that link-homotopy translates algebraically into this quotient. To convince ourselves, let us take a look at what the 'fully reduced' condition generates in the context of braids. Recall from Corollary 2.2.6 that the pure homotopy braid group hP_n is given by taking the reduced quotient of the pure braid group generated by the generators A_{ij} for $1 \leqslant i < j \leqslant n$. We consider first some relations of the fully reduced pure braid group $\mathcal{R}_{\mathcal{F}}P_n$.

Proposition 5.2.6. *If we set three indices $1 \leqslant r < i < j \leqslant n$ then the associated pure braids generators in $\mathcal{R}_{\mathcal{F}}P_n$ satisfy:*

$$[A_{ri},A_{rj}] = [A_{rj},A_{ij}] = [A_{ij},A_{ri}].$$

For any distinct pairs of indices $\{r,i\} \cap \{s,j\} = \varnothing$, in $\mathcal{R}_{\mathcal{F}}P_n$, we also have that:

$$[A_{ri},A_{sj}] = 1$$

Proof. The first relations are already true in $hP_n = \mathcal{R}P_n$, as mentioned in Theorem 2.2.6, so they must hold in $\mathcal{R}_{\mathcal{F}}P_n$. The other equality is also verified most of the time in $hP_n = \mathcal{R}P_n$, with the only remaining case being, without loss of generality, when $1 \leqslant r < s < i < j \leqslant n$. In that case, according to Theorem 2.2.6, we have:

$$[A_{ri},A_{sj}] = [[A_{ij},A_{rj}],A_{sj}],$$

which we rewrite using the inverse of the first relation as:

$$[A_{ri},A_{sj}] = [[A_{rj},A_{ri}],A_{sj}].$$

Then applying relation (1) and (3) from Proposition 5.2.4 as we did in the proof of Proposition 5.2.5 we get,

$$[A_{ri},A_{sj}] = [[A_{ri},A_{sj}],A_{rj}].$$

And finally we conclude using Theorem 2.2.6 again,

$$[A_{ri},A_{sj}] = [[[A_{ij},A_{rj}],A_{sj}],A_{rj}] = 1,$$

where the last equality holds since we have a commutator with repeats. □

Lemma 5.2.7. *The fully reduced pure braid group $\mathcal{R}_{\mathcal{F}}P_n$ is nilpotent of order* 3.

Proof. Set A_{ij}, A_{rs} and A_{kl} three generators in $\mathcal{R}_{\mathcal{F}}P_n$. We simply need to show that the commutator $C = [A_{kl},[A_{rs},A_{ij}]]$ is trivial in $\mathcal{R}_{\mathcal{F}}P_n$. First, according to the second equality from Proposition 5.2.6, the commutator $[A_{rs},A_{ij}]$ is trivial if $\{i,j\} \cap \{r,s\} = \varnothing$. Otherwise, we can suppose without loss of generality that $j = s$. We get then

$$C = [A_{kl},[A_{rj},A_{ij}]] = [A_{kl},[A_{ij},A_{ri}]] = [A_{kl},[A_{ri},A_{rj}]].$$

Then, using (2) from Proposition 5.2.4 we also have that

$$C^{-1} = [A_{rj},[A_{kl},A_{ij}]] = [A_{ij},[A_{kl},A_{ri}]] = [A_{ri},[A_{kl},A_{rj}]].$$

Now using the second relation from Proposition 5.2.6 again, we have that C is trivial if one of the following equalities hold

$$\{k,l\} \cap \{i,j\} = \varnothing, \quad \{k,l\} \cap \{r,i\} = \varnothing, \quad \{k,l\} \cap \{r,j\} = \varnothing.$$

If none of these equalities holds, then we have $\{k,l\} = \{i,j\}$, or $\{k,l\} = \{r,i\}$, or $\{k,l\} = \{r,j\}$ and C is also trivial. □

Theorem 5.2.8. *The fully reduced pure braid group $\mathcal{R}_{\mathcal{F}}P_n$ coincides with the third nilpotent quotient of the pure braid group.*

Proof. According to Lemma 5.2.7 we only need to prove that the commutator $[\omega,\lambda\omega\lambda^{-1}]$ belongs to the third subgroup of the lower central series. That is shown by the following computation:

$$[\omega,\lambda\omega\lambda^{-1}] = \omega\lambda\omega\lambda^{-1}\omega^{-1}\lambda\omega^{-1}\lambda^{-1} = \omega[\lambda,\omega]\omega^{-1}[\omega,\lambda] = [\omega,[\lambda,\omega]]$$

□

5.3 Link-homotopy using graph-claspers

In this section, we once again attempt to broaden the concept of link-homotopy within the context of homology cobordisms. We draw inspiration from the characterization in terms of repeated claspers (see Lemma 1.1.10). With this objective in mind, let us first define *graph-claspers* within the context of homology cylinders. To provide a rough comparison, claspers, as defined in Chapter 1, are distinguished from graph-claspers by their leaves: previously, they were disks intersecting tangle strands, whereas now they are framed knots.

5.3.1 Graph-claspers

Let M be a compact oriented 3-manifold.

Definition 5.3.1. *A connected surface G smoothly embedded in the interior of M is called a graph-clasper in M if it can be decomposed into leaves, nodes, and edges as follows:*

- ***Edges** are 1-handles that connect leaves and/or nodes, and each edge having two* 'ends'*, namely the attaching loci of the 1-handle.*
- ***Leaves** are framed knots, i.e., embeddings of annuli. Each leaf should have precisely one end of an edge attached to it.*
- ***Nodes** are discs, and each node should have exactly three ends of edges attached to it.*

When provided with a graph-clasper $G \subset M$, we can omit its leaves and collapse the remainder into a one-dimensional graph. This process results in a uni-trivalent graph known as the *shape* of G. Graph-claspers whose shape is a tree graph are called **tree-claspers**.

As before, we depict graph-claspers diagrammatically, as shown in Figure 5.5, for example. To

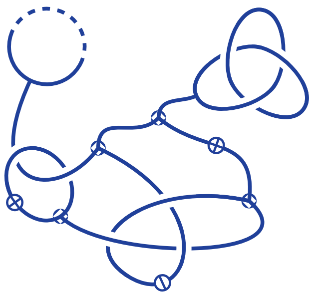


Figure 5.5: Diagram of a graph-clasper.

recover the represented graph-clasper, simply thicken the diagram using the blackboard framing convention. The nodes are represented by large dots and are thickened according to Figure 5.6. Additionally, we use markers called **twists** to indicate the presence of half-twists (see Figure 5.7).

Finally, we also use **boxes**, a graphical convention representing the entanglement of three leaves, as depicted in Figure 5.8.

Definition 5.3.2. *Let G be a graph-clasper in M. We define the* **degree** *of G, denoted by* $\deg(G)$*, as its number of nodes. Graph-claspers of degree* 0 *consist of only one edge and two leaves, see Figure 5.9.*

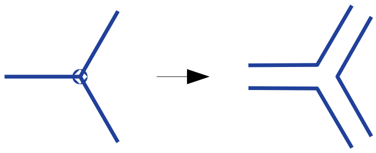


Figure 5.6: The diagrammatic node thickening pattern.

Figure 5.7: The diagrammatic negative and positive twist thickening patterns.

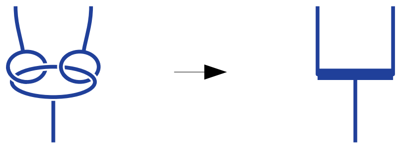


Figure 5.8: Leaf arrangement corresponding to a box.

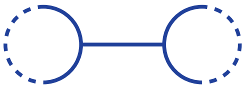


Figure 5.9: A degree 0 graph-clasper.

We stress that the notion of degree only makes sense for graph-claspers, which are connected surfaces decomposed into nodes, edges, and leaves. In particular, boxes can be misleading in this respect; they must be thought of as the junction of three claspers.

Given a disjoint union of graph-claspers F in M, there is a procedure called **surgery** detailed in [Hab00b] to construct a new manifold denoted M_F. First, we replace each node with three leaves forming a copy of the Borromean rings, as shown in Figure 5.10. This yields a union of degree 0

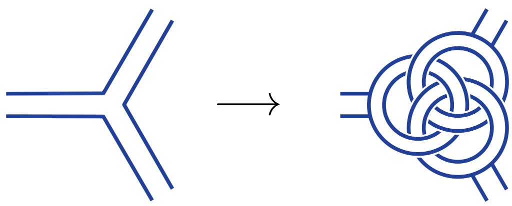


Figure 5.10: Replacing nodes by Borromean ring leaves.

graph-claspers. Next, we replace each degree-zero clasper with a two-component framed link, as shown in Figure 5.11. Finally, we apply Dehn surgery along the resulting framed link to obtain M_F.

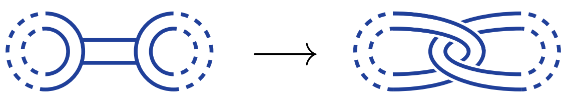


Figure 5.11: An example of a diagrammatic graph-clasper.

Let us recall some clasper calculus from [Hab00b].

Proposition 5.3.3. *[Hab00b, Proposition 2.7] The set of moves on graph-claspers depicted in Figure 5.12 yields surgery results that are isotopic.*

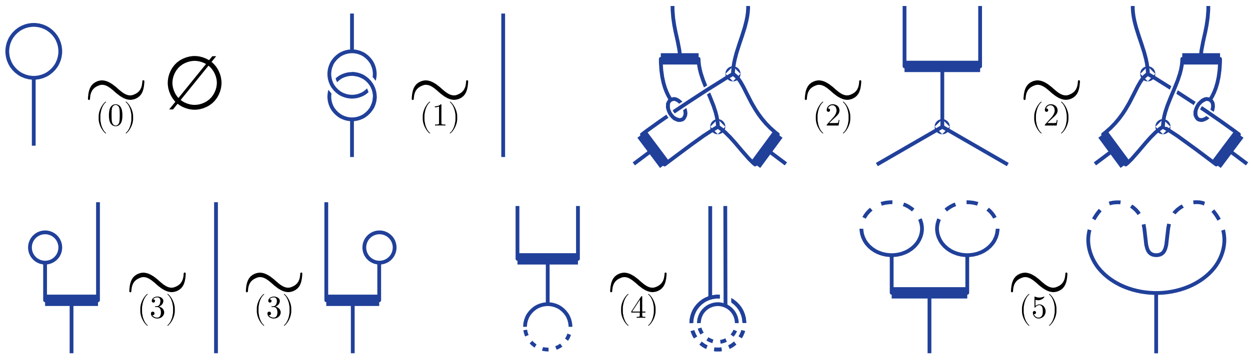


Figure 5.12: Some moves of clasper calculus.

Definition 5.3.4. *We call* $\mathbf{Y_k}$**-equivalence** *the equivalence relation on 3-manifolds generated by surgery on graph-claspers of degree at least k and ambient isotopies. We use the notation*

$$M \underset{Y_k}{\sim} M'$$

to mean that the two manifolds M and M' are Y_k-equivalent.

Surgery along a degree 1 graph-clasper, coincides with the *Borromean surgery* introduced by S. Matveev [Mat87]. It follows from the main result of [Mat87] that any integral homology sphere is Y_1-equivalent to S^3.

5.3.2 Homology cylinders and graph-claspers

We now delve back into the realm of homology cobordisms, specifically focusing on the set of homology cobordisms that are Y_k-equivalent to $\Sigma \times [-1,1]$, denoted as $\mathcal{C}_k(\Sigma)$. These sets are indeed submonoids of $\mathcal{C}(\Sigma)$ (see [Gou99, Hab00b]). Remarkably, as shown in [MM03, Section 4.1], the first of these submonoids coincides with the monoid of homology cylinders, i.e.,

$$\mathcal{C}_1(\Sigma) = \mathcal{IC}(\Sigma).$$

In other words any homology cylinders $C \in \mathcal{IC}(\Sigma)$ can be *presented* by a union of graph-claspers in $\Sigma \times [-1,1]$, meaning there exists a disjoint union of graph-clasper F in $\Sigma \times [-1,1]$ such that $C = (\Sigma \times [-1,1])_F$. Additionally, as for example stated in [HM12, Proposition 5.4], the resulting **Y-filtration**

$$\mathcal{C}(\Sigma) = \mathcal{C}_0(\Sigma) \supset \mathcal{C}_1(\Sigma) \supset \mathcal{C}_2(\Sigma) \supset \mathcal{C}_3(\Sigma) \supset \cdots$$

is finer than the Johnson filtration, in the sense that for any k,

$$\mathcal{C}_k(\Sigma) \subset \mathcal{C}(\Sigma)[k].$$

5.3.2.1 Link-homotopy for homology cylinders I

As mentioned in the introduction, link-homotopy is closely related to the notion of concordance. Indeed, it constitutes a more flexible equivalence relation. More precisely, if two links are concordant, then they are also link-homotopic. Therefore, in order to define a notion of link-homotopy for homology cobordisms, it seems natural to begin by examining the analogue of concordance for homology cylinders, and its interpretation in terms of graph-claspers

Definition 5.3.5. *Two homology cobordisms (C_1, i_1) and (C_2, i_2) over Σ are* **homology cobordant** *if the closed, oriented 3-manifold obtained by gluing C_1 and the reverse of C_2 (i.e., $C_1 \cup_{i_1 \circ i_2^{-1}} (-C_2)$) bounds a compact, oriented smooth 4-manifold W in such a way that both inclusions $C_1 \subset W$ and $C_2 \subset W$ induce homology isomorphisms. Here, $-C_2$ represents the homology cobordism given by reversing the orientation of C_2 together with the homeomorphism $i_2 \circ \tau$, where τ is the involution of $\Sigma \times [-1,1]$ defined by $\tau(x,t) = (x, -t)$.*

Being homology cobordant defines an equivalence relation among homology cobordisms, which is consistent with their composition. The resulting quotient monoid is known as the **homology cobordism group** and is denoted as $\mathcal{HC}(\Sigma)$ (see [GL05]). As the name suggests, this monoid forms a group, with the inverse of an element C given by $-C$. Moreover, by considering homology cylinders, we obtain a subgroup of $\mathcal{HC}(\Sigma)$ denoted as $\mathcal{HIC}(\Sigma)$. The *homology cobordism class* of a homology cobordism refers to its equivalence class as an element of the homology cobordism group.

Theorem 5.3.6. *[Lev01, Theorem 2] Surgery along graph-claspers that are not tree-claspers does not change the homology cobordism class of a homology cobordism.*

Remark 5.3.7. *Theorem 5.3.6 implies that in order to define a notion of link-homotopy which is consistent with the homology cobordism group, it is necessary that surgeries induced by graph-claspers that are not tree-claspers do not change the link-homotopy class of a homology cobordism.*

Let us fix $\mathfrak{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ a symplectic basis of the first homology group $H_1(\Sigma; \mathbf{Z})$ illustrated in Figure 5.13. We can see a leaf of a graph-clasper in $\Sigma \times [-1,1]$, as an element of

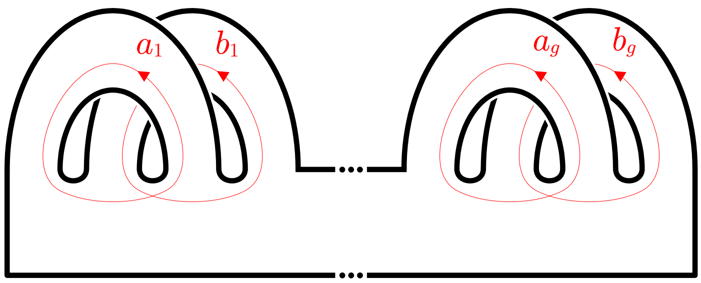


Figure 5.13: Symplectic basis $\mathfrak{B} = \{a_1, b_1, \ldots, a_g, b_g\}$ of $H_1(\Sigma; \mathbf{Z})$.

$H_1(\Sigma; \mathbf{Z})$. From this interpretation, we can define a first tentative notion of repeated graph-clasper.

Definition 5.3.8. *A graph-clasper G in $\Sigma \times [-1,1]$, has* ***repeats*** *if at least two of its leaves, seen as element of $H_1(\Sigma; \mathbf{Z})$, belong to the generators a_1 ,b_1, $\ldots$, a_g, b_g and are equal.*

Remark 5.3.9. *We could have proposed a larger definition of repetition, in which the leaves only need to represent the same element of $H_1(\Sigma; \mathbf{Z})$. However, we will not delve further as the seemingly finer notion of Definition 5.3.8 already proves unsatisfactory. Indeed, we show in Example 5.3.11 that the link-homotopy based on repeated claspers from Definition 5.3.8 corresponds almost to the Y_2-equivalence.*

Definition 5.3.10. *Let us consider two disjoint unions of graph-claspers F_1 and F_2 in $\Sigma \times [-1,1]$. Suppose that F_1 differs from F_2 by either graph-claspers that are not tree-claspers or tree-claspers with repeats. We say that the two homology cylinders given by $(\Sigma \times [-1,1])_{F_1}$ and $(\Sigma \times [-1,1])_{F_2}$ are* **link-homotopic***.*

Example 5.3.11. *The series of equivalences presented in Figure 5.14 demonstrates that the link-homotopy, as defined in Definition 5.3.10, nearly implies the Y_2-equivalence. To be more specific, any degree-2 clasper with one of its leaves matching one of the generators a_1, b_1, ..., a_g, b_g of $H_1(\Sigma; \mathbf{Z})$, is trivial up to link-homotopy. Let us take a closer look at these equivalences. The first pair of graph-*

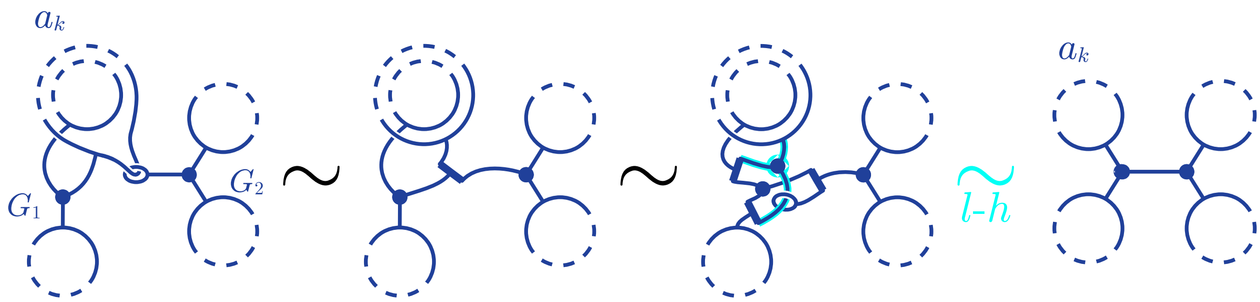


Figure 5.14: The link-homotopy from Definition 5.3.8 imply the Y_2-equivalence.

claspers, $G_1 \cup G_2$, is trivial up to link-homotopy. Indeed, the graph-clasper G_1 has repeats and can thus be deleted up to link-homotopy; this leaves us with the graph-clasper G_2, containing a leaf bounding a disk, which is thus trivial by moves (0) from Proposition 5.3.3. Subsequently, we apply more clasper calculus from Proposition 5.3.3. The first equivalence is given by moves (5) and (1), which introduce a box. The next equivalence involves applying move (2) to this box. Applying move (4) twice yields a new graph-clasper with repeats. Finally, we remove it up to link-homotopy and use move (3) to obtain the last equivalence.

Example 5.3.11 shows that any degree 2 tree-clasper, having a leaf representing a generator a_i or b_i for some i, can be deleted up to link-homotopy. Therefore, Definition 5.3.10 is not satisfactory, and we need to find a weaker definition of link-homotopy that provides better control over the nature of leaves with repeats.

5.3.2.2 Link-homotopy for homology cylinders II

The above tentative definitions of link-homotopy proved unsatisfactory, leading us to an even more constrained notion (see Definition 5.3.14). The latter is based on the *simplification of leaves* developed in [GGP01, Section 4.3]. Please note that the convention utilized in [GGP01] for surgery along a graph clasper, is the opposite to the convention used in this thesis.

Lemma 5.3.12. *[GGP01, Corollary 4.3] Let G be a graph-clasper of degree k in $\Sigma \times [-1{,}1]$, and let l be a leaf of G. An arc α starting at the external vertex incident to l and ending at another point on l splits l into two arcs, l_1 and l_2. Denote by G_1 and G_2 the graph-claspers obtained from G by replacing the leaf l with $l_1 \cup \alpha$ and $l_2 \cup \alpha$ respectively, see Figure 5.15. Then*

$$\big(\Sigma \times [-1{,}1]\big)_G \underset{Y_{k+1}}{\sim} \big(\Sigma \times [-1{,}1]\big)_{G_1} \cdot \big(\Sigma \times [-1{,}1]\big)_{G_2}.$$

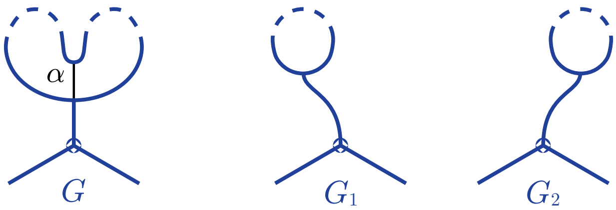


Figure 5.15: Graph-claspers G, G_1 and G_2 of Lemma 5.3.12.

The idea is to apply Lemma 5.3.12 and clasper calculus, in order to reexpress, up to higher degree graph-claspers, any disjoint union of tree-claspers in $\Sigma \times [-1{,}1]$ as a product of 'simpler' tree-claspers, with leaves of two specific types:

- **$\mathfrak{B}$-leaves**: leaves that are parallel copies of the curves a_i or b_i, framed along Σ and pushed inside $\Sigma \times [-1{,}1]$,
- **Special-leaves**: leaves which bound a disk disjoint from the rest of the tree-clasper and which are (-1)-framed.

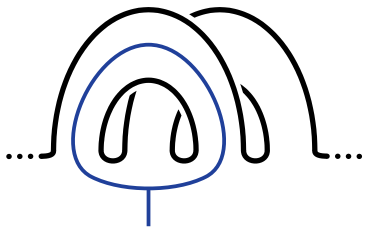


Figure 5.16: A $\mathfrak{B}$-leaf.

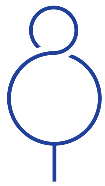


Figure 5.17: A special-leaf.

The fact that we can reduce the study to these two types of leaves only, follows from the same arguments as in [GGP01, Section 4.3].

Let us now define the notion of a *simple tree-clasper* using these two types of leaves. It is important to note that the term 'simple' also appears in [Hab00b], but we use it here in a different way.

Definition 5.3.13. *Given a disjoint union of graph-claspers F in $\Sigma \times [-1{,}1]$, a* **simple tree-clasper** *T, is a tree-clasper that lives in a 'slice' $\Sigma \times [-\epsilon{,}\epsilon]$ of $\Sigma \times [-1{,}1]$ which is disjoint from $G \backslash T$, and such that all of its leaves are either $\mathfrak{B}$-leaves or special-leaves.*

Note that, in particular, having repeats for a simple tree-clasper means that it contains two $\mathfrak{B}$-leaves that are parallel copies of the same curve.

Definition 5.3.14. *We define the* **link-homotopy** *relation between homology cylinders, presented by unions of graph-claspers in* $\Sigma \times [-1,1]$*, as the equivalence relation generated by surgeries on the three following types of graph-claspers:*

- *Graph claspers that are not trees,*
- *Graph claspers of degree at least* $2g-1$*,*
- *Simple tree-claspers having repeats.*

Let us discuss the three types of surgeries generating the link-homotopy relation in Definition 5.3.14.

Firstly, in accordance with Remark 5.3.7, surgeries on graph-claspers that are not tree-claspers must preserve homology cylinders up to link-homotopy.

Secondly, as mentioned earlier, the procedure of simplification of leaves works up to higher-degree tree-claspers. To ensure the termination of this procedure, it is necessary to eliminate all claspers beyond a certain degree. The degree $2g-1$, which corresponds to tree-claspers with $2g+1$ leaves, appears to be the suitable degree for this purpose. To justify this choice, we draw upon the analogy between string-links and homology cobordisms discussed at the beginning of Section 5.2. In the case of string-links with n components, C_n-equivalence implies link-homotopy: claspers with $n+1$ leaves inevitably have repeats and are thus trivial up to link-homotopy. Analogously, since $H_1(\Sigma; \mathbf{Z})$ has rank $2g$, it seems natural to eliminate all graph claspers with $2g+1$ leaves, which precisely corresponds to graph-claspers of degree at least $2g-1$.

Finally, we eliminate tree-claspers with repetitions once the clasper union is rewritten as a product of simple tree-claspers.

All the constraints discussed previously lead us to Definition 5.3.14, which, although somewhat unnatural, appears to be a promising candidate for a theory of link-homotopy for homology cylinders. We will not pursue this study further here, but consider this notion as a possible starting point for future research in this direction.

Remark 5.3.15. *The definition of link-homotopy in terms of simple tree-clasper, can probably be further refined. Indeed, such claspers containing a special leaf, can often be deleted up to higher order claspers; see [GGP01, Lemma 4.9]. As a matter of fact, the latter result, combined with the Slide move for special leaves [GGP01, Theorem 3.1], seem to suggest that only degree 1 graph-claspers with three special leaves would remain.*
The first Johnson homomorphism does not detect these particular tree-claspers, as shown in [MM03], which seems to conflict with the Milnor-Johnson correspondence. Indeed, Milnor string-link invariants provide a complete link-homotopy invariants.
This suggests a possible adjustment of Definition 5.3.14 making these degree 1 *tree-claspers trivial up to link-homotopy. However, this would further complicate the already involved Definition 5.3.14. An alternative would be to keep the definition unchanged, knowing that these tree-claspers are* 2*-torsion element and can be detected by the Rochlin invariant. In other words, we can group these terms together up to isotopy and eliminate them pairwise: parity is determined by the Rochlin invariant, as shown in [MM03].*

Appendix A

Code of the proof of Theorem 2.4.19

```
import itertools

#   The first part of the program is dedicated to the computation of the
# representation $\gamma$. In the following functions, the variable 'Commu' is a
# sequence [i1,...,im] representing the commutator (i1,...,in) in $\mathcal{V}$.
# The variable 'Index' is an integer representing the index of the homotopy braid
# generator $\sigma_i$.
#   The first function, IHX, serves as a preparatory function for the upcoming
# computation. Then the two functions Gamma_plus and Gamma_minus compute
# \gamma(\sigma_i)(i_1,...,im) and \gamma(\sigma_i^{-1})(i_1,...,im) and return a
# list of lists in the form [[coef1,I1],[coef2,I2],...,[coefm,Im]], corresponding
# to the linear combination coef1(I1)+coef2(I2)+...+coefm(Im) in $\mathcal{V}$.

def IHX(SubCommu,Commu):
    return([[Commu[0]+1,*Commu[1:Commu[0]],SubCommu[0],*Commu[Commu[0]:]],
           [*Commu[:Commu[0]+1],SubCommu[0],*Commu[Commu[0]+1:]]])

def Gamma_plus(Index,Commu):
    k=-1
    l=-1
    j=0
    while (k+1)*(l+1)==0 and j<len(Commu):
        if Commu[j]==Index :
            k=j
        if Commu[j]==Index+1:
            l=j
        j+=1
    if l==-1:
        if k==-1:
            return([[1]+Commu])
        return([[1,*Commu[:k],Index+1,*Commu[k+1:]]])
```

```
    if k==-1:
        Commu0=[*Commu[:l],Index,*Commu[l+1:]]
        Commu1=[*Commu[:l],Index,*Commu[l:]]
        if l>0:
            Commu2=[*Commu[:l+1],Index,*Commu[l+1:]]
            return([[1]+Commu0,[1]+Commu1,[-1]+Commu2])
        return([[1]+Commu0,[1]+Commu1])
    if k>0:
        if k<l:
            return([[1,*Commu[:k],Index+1,*Commu[k+1:l],Index,*Commu[l+1:]]])
        return([[1,*Commu[:l],Index,*Commu[l+1:k],Index+1,*Commu[k+1:]]])
    J=Commu[1:l]
    J.reverse()
    L=[[2,Index,*Commu[l:]]]
    while J!=[]:
        L=[IHX(J,K)[j] for K in L for j in range(0,2)]
        J=J[1:]
    for K in L:
        K[0]=(-1)**(K[0]+1)
    return(L)

def Gamma_minus(Index,Commu):
    k=-1
    l=-1
    j=0
    while (k+1)*(l+1)==0 and j<len(Commu):
        if Commu[j]==Index :
            k=j
        if Commu[j]==Index+1:
            l=j
        j+=1
    if k==-1:
        if l==-1:
            return([[1]+Commu])
        return([[1,*Commu[:l],Index,*Commu[l+1:]]])
    if l==-1:
        Commu0=[*Commu[:k],Index+1,*Commu[k+1:]]
        Commu1=[*Commu[:k+1],Index+1,*Commu[k+1:]]
        if k>0:
            Commu2=[*Commu[:k],Index+1,*Commu[k:]]
            return([[1]+Commu0,[1]+Commu1,[-1]+Commu2])
        return([[1]+Commu0,[1]+Commu1])
    if k>0:
        if k<l:
```

```
            return([[1,*Commu[:k],Index+1,*Commu[k+1:l],Index,*Commu[l+1:]]])
        return([[1,*Commu[:l],Index,*Commu[l+1:k],Index+1,*Commu[k+1:]]])
    J=Commu[1:l]
    J.reverse()
    L=[[2,Index,*Commu[l:]]]
    while J!=[]:
        L=[IHX(J,K)[j] for K in L for j in range(0,2)]
        J=J[1:]
    for K in L:
        K[0]=(-1)**(K[0]+1)
    return(L)


#   The next functions, Proj_gamma_generator and Proj_gamma, compute the
# representation on any linear combination of commutators in $\mathcal{V}$. More
# precisely, these functions compute the projection onto the subspace generated
# by commutators of length lower or egual to k. Furthermore, the second function,
# Proj_gamma, computes the representation on any homotopy braid $\beta=\sigma_{i1}
# \sigma_{i2}...\sigma_{im}$ encoded by the variable Braid=[i1,i2,...,im].

def Proj_gamma_generator(k,Index,Linear_combi):
    if Index>0:
        return([[Y[0]*I[0]]+Y[1:] for I in Linear_combi
               for Y in Gamma_plus(Index,I[1:])
               if len(I)<=k+1 if len(Y)<=k+1])
    if Index<0:
        return([[Y[0]*I[0]]+Y[1:] for I in Linear_combi
               for Y in Gamma_minus(-Index,I[1:])
               if len(I)<=k+1 if len(Y)<=k+1])
    return([[]])

def Proj_gamma(k,Braid,Linear_combi):
    Braid.reverse()
    length=len(Braid)
    for i in range(0,length):
        Linear_combi=Proj_gamma_generator(k,Braid[i],Linear_combi)
        j=0
        while j<len(Linear_combi):
            l=j+1
            while l<len(Linear_combi):
                if Linear_combi[j][1:]==Linear_combi[l][1:]:
                    Linear_combi[j][0]=Linear_combi[j][0]+Linear_combi[l][0]
                    Linear_combi.pop(l)
                    l-=1
```

```
                l+=1
            if Linear_combi[j][0]==0:
                Linear_combi.pop(j)
                j-=1
            j+=1
    return(Linear_combi)


# We now define the Inverse, Commutator, and Simplification functions to perform
# operations on homotopy braids. The first two produce inverses and commutators
# of braids, while the Simplification function simplifies pairs of trivial
# generators $\sigma_i\sigma_i^{-1}$.

def Inverse(Braid):
    length=len(Braid)
    return([-Braid[length-i-1] for i in range(0,length)])

def Commutator(Braid1,Braid2):
    return(Braid1+Braid2+Inverse(Braid1)+Inverse(Braid2))

def Simplification(Braid):
    i=0
    while i<len(Braid)-1:
        if Braid[i]==-Braid[i+1]:
            Braid.pop(i+1)
            Braid.pop(i)
            i-=2
        i+=1


#   The function Comb_clasper_generator constructs the comb-clasper (i, j) as a
# word in the homotopy braid generators $\sigma_i$. Similarly, the function
# Comb_clasper constructs the comb-clasper (i1, i2, ..., in) as a word in the
# homotopy braid generators $\sigma_i$.

def Comb_clasper_generator(i,j):
    return([j-k for k in range(1,j-i)]+[i,i]+[-i-k for k in range(1,j-i)])

def Comb_clasper(I):
    length=len(I)-1
    T=Comb_clasper_generator(I[0],I[length])
    for i in I[1:length]:
        T=Commutator(T,Comb_clasper_generator(i,I[length]))
    return(T)
```

```
#  In the final part of the program, we construct the family {\theta_k} of
# homotopy braids from Section 2.4.2. We begin with the Test, Lambda_action, and
# Filter functions, which, given a linear combination of commutators in
# \mathcal{V}, allow us to retain only those corresponding to comb-claspers in
# nice position. Subsequently, we end with the Torsion_candidate function,
# explicitly computing the braid \theta_{p-2} from Section 2.4.2. More precisely,
# the Torsion_candidate takes a prime number p as input and returns the braid
# \theta_{p-2} along with its image under the gamma representation. It is worth
# noting that if this image contains the element [coef, 1, 2, ..., p], and coef
# is not divisible by p, then it provides an obstruction to the presence of
# torsion in the homotopy braid group.

def Test(Commu,Orbit):
    for Representative in Orbit:
        if Commu==Representative:
            return(1)
    return(0)

def Lambda_action(Sequence):
    S=[0]+Sequence[:-1]
    return([j+Sequence[-1]-Sequence[-2] for j in S])

def Filter(Linear_combi):
    i=0
    while i<len(Linear_combi):
        T=sorted(Linear_combi[i][1:])
        Orbit=[]
        for j in T[:-1]:
            T=Lambda_action(T)
            Orbit=Orbit+[T]
        m=1
        j=i+1
        while j<len(Linear_combi):
            if Test(sorted(Linear_combi[j][1:]),Orbit)==1:
                Linear_combi.pop(j)
                j-=1
                m+=1
            j+=1
        i+=1

def Torsion_candidate(p):
    Theta=['lambda']
```

```
Braid=[-i for i in range(1,p)]
for k in range(2,p):
    Braid_power_p=Braid*p
    Simplification(Braid_power_p)
    Image=Proj_gamma(k,Braid_power_p,[[1,p]])[1:]
    Filter(Image)
    for T in Image:
        Theta=[T]+Theta
        if T[0]>0:
            Braid=T[0]*Comb_clasper(T[1:])+Braid
        if T[0]<0:
            Braid=-T[0]*Inverse(Comb_clasper(T[1:]))+Braid
Braid_power_p=Braid*p
Simplification(Braid_power_p)
Image=Proj_gamma(p,Braid_power_p,[[1,p]])[1:]
return(Theta,Image)
```

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